

CATEGORICAL RESOLUTIONS, POSET SCHEMES AND DU BOIS SINGULARITIES

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ABSTRACT. We introduce the notion of a poset scheme and study the categories of quasi-coherent sheaves on such spaces. We then show that smooth poset schemes may be used to obtain categorical resolutions of singularities for usual singular schemes. We prove that a singular variety X possesses such a resolution if and only if X has Du Bois singularities. Finally we show that the de Rham-Du Bois complex for an algebraic variety Y may be defined using any smooth poset scheme which satisfies the descent over Y in the classical topology.

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1. INTRODUCTION

1.1. Categorical resolutions. There is a good notion of smoothness for a DG algebra A . Namely, A is called *smooth* if it is a perfect DG $A^{\text{op}} \otimes A$ -module. This notion is Morita invariant: if DG algebras A and B are derived equivalent (i.e. there exists a DG $A^{\text{op}} \otimes B$ -module M , such that the functor $(-)^{\mathbf{L}} \otimes_A M : D(A) \rightarrow D(B)$ is an equivalence), then A is smooth if and only if B is such. This allows one to define smoothness of derived categories $D(A)$, and consequently of cocomplete triangulated categories which possess a compact generator (and have an enhancement). Examples of such categories are the derived categories $D(X)$ of quasi-coherent sheaves on quasi-compact and separated schemes X (see for example [BoVdB]). The scheme X is smooth if and only if the category $D(X)$ is smooth in the above sense.

In the paper [Lu2] we have introduced the concept of a categorical resolution of singularities. Namely, given a DG algebra A , a categorical resolution of $D(A)$ is a pair (B, M) , where B is a smooth DG algebra and M is a DG $A^{\text{op}} \otimes B$ -module, such that the functor $(-)^{\mathbf{L}} \otimes_A M : D(A) \rightarrow D(B)$ is full and faithful on the subcategory of perfect DG A -modules. The main result of [Lu2] is the following theorem.

Theorem 1.1. *Let X be a separated scheme of finite type over a perfect field k . Then*

a) *There exists a classical generator $E \in D^b(\text{coh} X)$, such that the DG algebra $A = \mathbf{R}\text{Hom}(E, E)$ is smooth and hence the functor*

$$\mathbf{R}\text{Hom}(E, -) : D(X) \rightarrow D(A)$$

is a categorical resolution.

b) *Given any other classical generator $E' \in D^b(\text{coh} X)$ with $A' = \mathbf{R}\text{Hom}(E', E')$, the DG algebras A and A' are derived equivalent.*

This theorem provides an intrinsic categorical resolution for $D(X)$. This resolution has the flavor of Koszul duality. The resolving DG algebra A is Morita equivalent to its opposite A^{op} and usually has unbounded cohomology.

Example 1.2. *If in Theorem 1.1 $X = \text{Spec}(k[\epsilon]/\epsilon^2)$, and $E = k$, then $A = k[t]$, where $\deg(t) = 1$.*

We should note that the notion of categorical resolutions is different from the usual resolution of singularities. Namely if X is an algebraic variety and $\sigma : \tilde{X} \rightarrow X$ is its resolution of singularities, then $\mathbf{L}\sigma^* : D(X) \rightarrow D(\tilde{X})$ is a categorical resolution if and only if X has rational singularities. If $D(X) \rightarrow D(A)$ is a categorical resolution (and the singularities of X are not rational), we find that the category $D(A)$ has a closer relation to $D(X)$ than $D(\tilde{X})$. Also one may consider categorical resolutions of nonreduced schemes.

Conjecture. Let X be a separated scheme of finite type over a field. Then there exists a smooth DG algebra A with $H^i(A) = 0$ for $|i| > 0$ and a functor $D(X) \rightarrow D(A)$ which is a categorical resolution.

1.2. Smooth poset schemes and Du Bois singularities. In this article we introduce a new class of smooth categories, which are constructed by "gluing" the categories $D(X)$ for smooth schemes X . Namely, we consider *poset schemes* \mathcal{X} which by definition are diagrams of schemes $\{X_\alpha\}_{\alpha \in S}$ indexed by elements of a finite poset S with a morphism $f_{\alpha\beta} : X_\alpha \rightarrow X_\beta$ iff $\alpha \geq \beta$. There is a natural notion of a quasi-coherent sheaf on \mathcal{X} , which gives us the abelian category $Q\text{coh}\mathcal{X}$ and its derived category $D(\mathcal{X})$. This derived category is cocomplete and has a compact generator (if all schemes X_α are separated and quasi-compact). So $D(\mathcal{X}) \simeq D(A)$ for a DG algebra A . The category $D(\mathcal{X})$ is smooth if the poset scheme \mathcal{X} is smooth (i.e. all schemes X_α are such). In any case the category $D(\mathcal{X})$ has a natural semi-orthogonal decomposition with semi-orthogonal summands $D(X_\alpha), \alpha \in S$. In this last sense we consider $D(\mathcal{X})$ as a gluing of the categories $D(X_\alpha)$ along the morphisms $f_{\alpha\beta}$.

There is a natural notion of a morphism $\pi : \mathcal{X} \rightarrow X$ from a poset scheme \mathcal{X} to a scheme X and the corresponding functor $\mathbf{L}\pi^* : D(X) \rightarrow D(\mathcal{X})$. We say that π is a categorical resolution if \mathcal{X} is smooth and $\mathbf{L}\pi^*$ is a categorical resolution. We prove the following theorem (=Theorem 12.11).

Theorem 1.3. *Let X be a reduced separated scheme of finite type over a field of characteristic zero. Then X has a categorical resolution by a smooth poset scheme if and only if X has Du Bois singularities.*

The "if" direction in the theorem is essentially the definition of Du Bois singularities (plus the work [LNM1335]), and the other direction is a consequence of the general functorial formalism which we develop. This theorem proves the above conjecture in the case of Du Bois singularities.

Corollary 1.4. *Let X be a reduced separated scheme of finite type over a field of characteristic zero. Assume that X has Du Bois singularities. Then there exists a smooth DG algebra A and a categorical resolution $D(X) \rightarrow D(A)$, such that*

- 1) $H^i(A) = 0$ for $|i| \gg 0$;
- 2) $D(A)$ has a finite semi-orthogonal decomposition with summands $D(X_i)$ where each X_i is smooth and X_1 is a usual resolution of X ;
- 3) If X is proper, then each X_i is also proper. In particular in this case the DG algebra A is proper (has finite dimensional cohomology).

Theorems 10.1, 11.2, 11.5, 12.11, 14.1 may be viewed applications of our theory of smooth projective poset schemes to the study of Du Bois singularities. In particular, Theorem 14.1 asserts that the de Rham-Du Bois complex may be defined by means of *any* smooth projective poset scheme which satisfies the descent in the classical topology.

Our poset schemes are generalizations of *configuration schemes* studied in [Lu1]. (A configuration scheme is a poset scheme where all the structure morphisms $f_{\alpha\beta}$ are closed embeddings). Although the notion of a categorical resolution is not present explicitly in [Lu1] the ideas discussed in that paper are similar to what we do here.

1.3. Organization of the paper. The paper consists of two parts. In the first one we develop in detail the theory of poset schemes and discuss their relationship with categorical resolutions. In the second part we prove three results on degeneration of spectral sequences for smooth projective poset schemes (Theorems 10.1, 11.2, 11.5) These results are used to prove Theorem 14.1. In Theorem 12.11 we establish a connection between Du Bois singularities and the existence of a categorical resolution by a smooth poset scheme.

The appendix contains some general facts on functors between derived categories of quasi-coherent sheaves.

In [Lu2] we have collected some well known general categorical facts about cocomplete triangulated categories, existence of compact generators, smoothness of DG algebras, existence of enough h-injectives in derived categories of Grothendieck abelian categories, etc. These facts are not discussed in this article and we refer the reader to [Lu2] as needed.

I want to thank Tony Pantev who first suggested a connection between categorical resolutions by poset schemes and Du Bois singularities. A discussion of Du Bois singularities with Karl Schwede helped me understand the subject. Finally I thank the participants of algebraic geometry seminar in Steklov Institute in Moscow for their interest in this work.

Part 1. Categorical resolutions by poset schemes

2. QUASI-COHERENT SHEAVES ON POSET SCHEMES

We fix a base field k . A "scheme" means a separated quasi-compact k -scheme, all morphisms of schemes are assumed to be separated and quasi-compact. All the products and tensor products are taken over k unless specified otherwise. Throughout this article a "poset" (=a partially ordered set) means a *finite* poset.

Definition 2.1. Let $S = \{\alpha, \beta, \dots\}$ be a poset which we consider as a category: the set $\text{Hom}(\alpha, \beta)$ has a unique element if $\alpha \geq \beta$ and is empty otherwise. Then an S -scheme, or an S -poset scheme, or a poset scheme is simply a functor from S to the category of schemes. In other words, a poset scheme is a collection $\mathcal{X} = \{X_\alpha, f_{\alpha\beta}\}_{\alpha \geq \beta \in S}$, where X_α is a scheme and $f_{\alpha\beta} : X_\alpha \rightarrow X_\beta$ is a morphism of schemes, such that $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$. We call \mathcal{X} noetherian, regular, smooth, of finite type, essentially of finite type, etc. if all schemes $X_\alpha \in \mathcal{X}$ are such.

Definition 2.2. Let $\mathcal{X} = \{X_\alpha, f_{\alpha\beta}\}$ be a poset scheme. A quasi-coherent sheaf on \mathcal{X} is a collection $F = \{F_\alpha \in \text{Qcoh}(X_\alpha), \varphi_{\alpha\beta} : f_{\alpha\beta}^* F_\beta \rightarrow F_\alpha\}$ so that the morphisms φ satisfy the usual cocycle condition: $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \cdot f_{\alpha\beta}^* (\varphi_{\beta\gamma})$. Quasi-coherent sheaves on \mathcal{X} form a category in the obvious way. We denote this category $\text{Qcoh}\mathcal{X}$.

Lemma 2.3. The category $\text{Qcoh}\mathcal{X}$ is an abelian category.

Proof. Indeed, given a morphism $g : F \rightarrow G$ in $\text{Qcoh}\mathcal{X}$ we define $\text{Ker}(g)$ and $\text{Coker}(g)$ componentwise. Namely, put $\text{Ker}(g)_\alpha := \text{Ker}(g_\alpha)$, $\text{Coker}(g)_\alpha := \text{Coker}(g_\alpha)$. Note that $\text{Coker}(g)$ is well defined since the functors $f_{\alpha\beta}^*$ are right-exact. \square

Remark 2.4. A quasi-coherent sheaf F on a poset scheme $\mathcal{X} = \{X_\alpha, f_{\alpha\beta}\}$ can be equivalently defined as a collection $F = \{F_\alpha \in \text{Qcoh}(X_\alpha), \psi_{\alpha\beta} : F_\beta \rightarrow f_{\alpha\beta*} F_\alpha\}$, so that the morphisms ψ satisfy the usual cocycle condition: $\psi_{\alpha\gamma} = f_{\beta\gamma*}(\psi_{\alpha\beta}) \cdot \psi_{\beta\gamma}$.

Definition 2.5. *The quasi-coherent sheaf $\mathcal{O}_{\mathcal{X}} = \{\mathcal{O}_{X_\alpha}, \phi_{\alpha\beta} = \text{id}\}$ is called the structure sheaf of \mathcal{X} . Also for each $i \geq 0$ we have the natural sheaf $\Omega_{\mathcal{X}}^i$ - the i -th exterior power of the sheaf of Kahler differentials $\Omega_{\mathcal{X}}^1$. Together these sheaves form the deRham complex $\Omega_{\mathcal{X}}^\bullet$ (as usual the differential in $\Omega_{\mathcal{X}}^\bullet$ is not $\mathcal{O}_{\mathcal{X}}$ -linear; it is a differential operator of order 1).*

2.1. Operations with quasi-coherent sheaves on poset schemes. Let S be a finite poset and \mathcal{X} be an S -scheme. Denote for short $\mathcal{M} = Qcoh\mathcal{X}$ and $\mathcal{M}_\alpha = QcohX_\alpha$. For $F \in \mathcal{M}$ define its support $\text{Supp}(F) = \{\alpha \in S | F_\alpha \neq 0\}$.

Define a topology on S by taking as a basis of open sets the subsets $U_\alpha = \{\beta \in S | \beta \geq \alpha\}$.

Note that $Z_\alpha = \{\gamma \in S | \gamma \leq \alpha\}$ is a closed subset in S .

Let $U \subset S$ be open and $Z = S - U$ - the complementary closed. Let \mathcal{M}_U (resp. \mathcal{M}_Z) be the full subcategory of \mathcal{M} consisting of objects F with support in U (resp. in Z). For every object F in \mathcal{M} there is a natural short exact sequence

$$0 \rightarrow F_U \rightarrow F \rightarrow F_Z \rightarrow 0,$$

where $F_U \in \mathcal{M}_U$, $F_Z \in \mathcal{M}_Z$. Indeed, take

$$(F_U)_\alpha = \begin{cases} F_\alpha, & \text{if } \alpha \in U, \\ 0, & \text{if } \alpha \in Z. \end{cases}$$

$$(F_Z)_\alpha = \begin{cases} F_\alpha, & \text{if } \alpha \in Z, \\ 0, & \text{if } \alpha \in U. \end{cases}$$

We may consider U (resp. Z) as a subcategory of S and restrict the poset scheme \mathcal{X} to U (resp. to Z). Denote these restrictions by $\mathcal{X}(U)$ and $\mathcal{X}(Z)$ and the corresponding categories by $\mathcal{M}(U)$ and $\mathcal{M}(Z)$ respectively.

Denote by $j : U \hookrightarrow S$ and $i : Z \hookrightarrow S$ the inclusions. We get the obvious restriction functors

$$j^* = j^! : \mathcal{M} \rightarrow \mathcal{M}(U), \quad i^* : \mathcal{M} \rightarrow \mathcal{M}(Z).$$

Clearly these functors are exact. The functor j^* has an exact left adjoint $j_! : \mathcal{M}(U) \rightarrow \mathcal{M}$ (“extension by zero”). Its image is the subcategory \mathcal{M}_U . The functor i^* has an exact right adjoint $i_* = i_! : \mathcal{M}(Z) \rightarrow \mathcal{M}$ (also “extension by zero”). Its image is the subcategory \mathcal{M}_Z . It follows that j^* and i_* preserve injectives (as right adjoints to exact functors). We have $j^*j_! = \text{Id}$, $i^*i_* = \text{Id}$.

Note that the short exact sequence above is just

$$0 \rightarrow j_!j^*F \rightarrow F \rightarrow i_*i^*F \rightarrow 0,$$

where the two middle arrows are the adjunction maps.

The functor i_* also has a left-exact right adjoint functor $i^!$. Namely $i^!F$ is the largest subobject of F which is supported on Z .

For $\alpha \in S$ denote by $j_\alpha : \{\alpha\} \hookrightarrow S$ the inclusion. The inverse image functor $j_\alpha^* : \mathcal{M} \rightarrow \mathcal{M}_\alpha$, $F \mapsto F_\alpha$ has a right-exact left adjoint $j_{\alpha+}$ defined as follows

$$(j_{\alpha+}P)_\beta = \begin{cases} f_{\beta\alpha}^*P, & \text{if } \beta \geq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $P \in \mathcal{M}_\alpha$, $\text{Supp } j_{\alpha+}P \subset U_\alpha$.

We also consider the “extension by zero” functor $j_{\alpha!} : \mathcal{M}_\alpha \rightarrow \mathcal{M}$ defined by

$$j_{\alpha!}(P)_\beta = \begin{cases} P, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.6. *The functor $j_\alpha^* : \mathcal{M} \rightarrow \mathcal{M}_\alpha$ has a right adjoint $j_{\alpha*}$. This functor $j_{\alpha*}$ is left-exact and preserves injectives. For $P \in \mathcal{M}_\alpha$ $\text{Supp}(j_{\alpha*}P) \subset Z_\alpha$.*

Proof. Given $P \in \mathcal{M}_\alpha$ we set

$$j_{\alpha*}(P)_\gamma = \begin{cases} f_{\alpha\gamma*}(P), & \text{if } \gamma \leq \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

and the structure map

$$\varphi_{\gamma\delta} : f_{\gamma\delta}^*((j_{\alpha*}P)_\delta) \rightarrow (j_{\alpha*}P)_\gamma$$

is the adjunction map

$$f_{\gamma\delta}^* f_{\alpha\gamma*} P = f_{\gamma\delta}^* f_{\gamma\delta*} f_{\alpha\gamma*} P \rightarrow f_{\alpha\delta*} P$$

if $\delta \leq \gamma \leq \alpha$ and $\varphi_{\gamma\delta} = 0$ otherwise.

It is clear that $j_{\alpha*}$ is left-exact and that $\text{Supp}(j_{\alpha*}P) \subset Z_\alpha$.

Let us prove that $j_{\alpha*}$ is the right adjoint to j_α^* .

Let $P \in \mathcal{M}_\alpha$ and $M = \{M_\gamma, \varphi_{\gamma\beta}\} \in \mathcal{M}$. Given $g_\alpha \in \text{Hom}(M_\alpha, P)$ for each $\gamma \leq \alpha$ we obtain a map $g_\alpha \cdot \varphi_{\alpha\gamma} : f_{\alpha\gamma}^* M_\gamma \rightarrow P$ and hence by adjunction $g_\gamma : M_\gamma \rightarrow f_{\alpha\gamma*} P = (j_{\alpha*}P)_\gamma$. The collection $g = \{g_\gamma\}$ is a morphism $g : M \rightarrow j_{\alpha*}P$. It remains to show that the constructed map

$$\text{Hom}(M_\alpha, P) \rightarrow \text{Hom}(M, j_{\alpha*}P)$$

is surjective or, equivalently, that the restriction map

$$\text{Hom}(M, j_{\alpha*}P) \rightarrow \text{Hom}(M_\alpha, P), \quad g \mapsto g_\alpha$$

is injective.

Assume that $0 \neq g \in \text{Hom}(M, j_{\alpha*}P)$, i.e. $g_\gamma \neq 0$ for some $\gamma \leq \alpha$. By definition we have the commutative diagram

$$\begin{array}{ccc} f_{\alpha\gamma}^* M_\gamma & \xrightarrow{f_{\alpha\gamma}^*(g_\gamma)} & f_{\alpha\gamma}^* f_{\alpha\gamma*} P \\ \varphi_{\alpha\gamma} \downarrow & & \downarrow \epsilon_P \\ M_\alpha & \xrightarrow{g_\alpha} & P, \end{array}$$

where ϵ_P is the adjunction morphism. Note that $\epsilon_P f_{\alpha\gamma}^*(g_\gamma) : f_{\alpha\gamma}^* M_\gamma \rightarrow P$ is the morphism, which corresponds to $g_\gamma : M_\gamma \rightarrow f_{\alpha\gamma*} P$ by the adjunction property. Hence $\epsilon_P f_{\alpha\gamma}^*(g_\gamma) \neq 0$. Therefore $g_\alpha \neq 0$. This shows the injectivity of the restriction map $g \mapsto g_\alpha$ and proves that $j_{\alpha*}$ is the right adjoint to j_α^* . Finally, $j_{\alpha*}$ preserves injectives being the right adjoint to an exact functor. \square

Lemma 2.7. *The abelian category \mathcal{M} is a Grothendieck category. In particular it has enough injectives and the corresponding category of complexes $C(\mathcal{M})$ has enough h-injectives [KaSch], Thm.14.1.7.*

Proof. For a usual quasi-compact and quasi-separated scheme X the category $Qcoh X$ is known to be Grothendieck [ThTr], Appendix B. The category \mathcal{M} is abelian 2.3 and has arbitrary direct sums (since the "gluing" functors $f_{\alpha\beta}^*$ preserve direct sums), so it has arbitrary colimits. Filtered colimits are exact, because the exactness is determined locally on each X_α . It remains to prove the existence of a generator for the abelian category \mathcal{M} . For each $\alpha \in S$ choose a generator $M_\alpha \in Qcoh X_\alpha$. We claim that $M := \oplus_\alpha (j_{\alpha+} M_\alpha)$ is a generator in \mathcal{M} . Indeed, let $g : F \rightarrow G$ be a morphism in \mathcal{M} , such that $g_* : \text{Hom}(M, F) \rightarrow \text{Hom}(M, G)$ is an isomorphism. We have

$$\text{Hom}(M, -) = \oplus_\alpha \text{Hom}(j_{\alpha+} M_\alpha, -) = \oplus_\alpha \text{Hom}(M_\alpha, (-)_\alpha).$$

So for each α the map $g_{\alpha*} : \text{Hom}(M_\alpha, F_\alpha) \rightarrow \text{Hom}(M_\alpha, G_\alpha)$ is an isomorphism, hence g_α is an isomorphism. Thus g is an isomorphism. \square

2.2. Summary of functors and their properties. For reader's convenience we list all the functors introduced so far together with their properties.

Functors: $j^* = j^!, j_!, i^*, i_* = i_!, i^!, j_\alpha^*, j_{\alpha+}, j_{\alpha*}$.

Exactness: $j^*, j_!, i^*, i_*, j_\alpha^*$ - exact; $i^!, j_{\alpha*}$ - left-exact; $j_{\alpha+}$ - right-exact.

Adjunction: $(j_!, j^*), (i^*, i_*), (i_*, i^!), (j_{\alpha+}, j_\alpha^*), (j_\alpha^*, j_{\alpha*})$ are adjoint pairs.

Preserve direct sums: All the above functors preserve direct sums. (The functor $j_{\alpha*}$ preserves direct sums because the morphisms $f_{\alpha\beta}$ are quasi-compact.)

Preserve injectives: $j^*, i_*, i^!, j_{\alpha*}$ preserve injectives because they are right adjoint to exact functors.

Tensor product: The bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ is defined componentwise: $(F \otimes G)_\alpha = F_\alpha \otimes_{\mathcal{O}_{X_\alpha}} G_\alpha$.

2.3. Cohomological dimension of poset schemes. We keep the notation of Subsection 2.1

Proposition 2.8. *If the poset scheme \mathcal{X} is regular noetherian, then \mathcal{M} has finite cohomological dimension.*

Proof. The proposition asserts that any F in \mathcal{M} has a finite injective resolution. Equivalently, a finite complex in \mathcal{M} is quasi-isomorphic to a finite complex of injectives. We argue by induction on the cardinality of S , the case $|S| = 1$ is well known.

Let $\beta \in S$ be a biggest element. Put $U = U_\beta = \{\beta\}$, $Z = S - U$. Let $j = j_\beta : U \hookrightarrow S$ and $i : Z \hookrightarrow S$ be the corresponding open and closed embeddings.

Fix F in \mathcal{M} ; it suffices to find finite injective resolutions for $j_! j^* F$ and $i_* i^* F$. Let $j^* F \rightarrow I_1$, $i^* F \rightarrow I_2$ be such resolutions in categories $\mathcal{M}(U)$ and $\mathcal{M}(Z)$ respectively. Then $i_* i^* F \rightarrow i_* I_2$ will be an injective resolution in \mathcal{M} . Note that $j_* I_1$ is a (finite) complex of injectives in \mathcal{M} and that the cone K of the natural morphism $j_* j^* F \rightarrow j_* I_1$ is acyclic on X_β . Hence by the induction assumption K is quasi-isomorphic to $i_* J$, where J is a finite complex of injectives in $\mathcal{M}(Z)$. Therefore the object $j_* j^* F$ has a finite injective resolution in \mathcal{M} .

Consider the short exact sequence

$$0 \rightarrow j_! j^* F \rightarrow j_* j^* F \rightarrow G \rightarrow 0.$$

Then $\text{Supp}(G) \subset Z$ and so by induction $G = i_* i^* G$ has a finite injective resolution in \mathcal{M} . Therefore the same is true for $j_! j^* F$. \square

3. DERIVED CATEGORIES OF POSET SCHEMES

Let S be a poset, \mathcal{X} an S -scheme, $\mathcal{M} = Qcoh \mathcal{X}$, $C(\mathcal{X}) = C(\mathcal{M})$ - the abelian category of complexes in \mathcal{M} , $Ho(\mathcal{X}) = Ho(\mathcal{M})$, $D(\mathcal{X}) = D(\mathcal{M})$ - its homotopy and derived category.

Let $U \xrightarrow{j} S \xleftarrow{i} Z$ be embeddings of an open U and a complementary closed Z . The exact functors $j^*, j_!, i^*, i_*, j_\alpha^*$ extend trivially to corresponding functors between derived categories $D(\mathcal{M})$, $D(\mathcal{M}(U))$, $D(\mathcal{M}(Z))$, $D(X_\alpha)$. To define the derived functors of the other functors we need h-injective and h-flat objects in $C(\mathcal{M})$. (There are enough h-injectives by Lemma 2.7)

Definition 3.1. *An object $F \in C(\mathcal{M})$ is called h-flat if for any acyclic complex $S \in C(\mathcal{M})$ the complex $F \otimes S$ is acyclic.*

Notice that for any $\alpha \in S$ the functor $j_{\alpha*} : C(X_\alpha) \rightarrow C(\mathcal{X})$ preserves h-injectives. Indeed, its left adjoint functor j_α^* preserves acyclic complexes. Denote by $SI(\mathcal{X}) \subset Ho(\mathcal{X})$ the full triangulated subcategory classically generated by objects $j_{\alpha*}M$, for h-injective $M \in C(X_\alpha)$. We call objects of $SI(\mathcal{X})$ *special h-injectives*. It is sometimes convenient to use the following lemma.

Lemma 3.2. *There are enough special injectives in $D(\mathcal{X})$.*

Proof. Fix $F \in C(\mathcal{X})$ and let $\beta \in S$ be a biggest element such that the complex F_β is not acyclic. Choose an h-injective resolution $\rho : F_\beta \rightarrow I$ in $D(X_\beta)$. By adjunction it induces a morphism $\sigma : F \rightarrow j_{\beta*}I$. By construction the cone C_σ of the morphism σ is acyclic on X_γ for all $\gamma \geq \beta$. So by induction we may assume that there exists a special h-injective J and a quasi-isomorphism $C_\sigma \rightarrow J$. So F is quasi-isomorphic to the (shifted) cone of a morphism $j_{\beta*}I \rightarrow J$. \square

It is known that for any quasi-compact separated scheme X there are enough h-flats in $D(X)$ [AlJeLi], Proposition 1.1. Clearly, an object $F \in C(\mathcal{X})$ is h-flat if and only if $F_\alpha \in C(X_\alpha)$ is h-flat for every $\alpha \in S$. Let $M \in C(X_\alpha)$ be h-flat. Then $j_{\alpha+}M \in C(\mathcal{X})$ is also such. Indeed, the inverse image functors $f_{\beta\alpha}^*$ preserve h-flats [Sp], Proposition 5.4. Denote by $SF(\mathcal{X}) \subset Ho(\mathcal{X})$ the full triangulated subcategory classically generated by objects $j_{\alpha+}M$, where $M \in C(X_\alpha)$ is h-flat. We call objects of $SF(\mathcal{X})$ *special h-flats*.

Lemma 3.3. *There are enough special h-flats in $D(\mathcal{X})$.*

Proof. Similar to the proof of Lemma 3.2 but using the adjoint pair $(j_{\alpha+}, j_\alpha^*)$ instead of $(j_\alpha^*, j_{\alpha*})$. \square

We now use h-injectives to define the right derived functors

$$\mathbf{R}j_{\alpha*} : D(X_\alpha) \rightarrow D(\mathcal{X}), \quad \mathbf{R}i^! : D(\mathcal{X}) \rightarrow D(\mathcal{X}(Z)),$$

and h-flats to define the left derived functor

$$\mathbf{L}j_{\alpha+} : D(X_\alpha) \rightarrow D(\mathcal{X})$$

and the derived functor $(-) \overset{\mathbf{L}}{\otimes} (-) : D(\mathcal{X}) \times D(\mathcal{X}) \rightarrow D(\mathcal{X})$ (by resolving any of the two variables).

3.1. Summary of functors and their properties.

Preserve h-flats and h-injectives: The functors $j^*, j_!, i^*, i_*, j_\alpha^*, j_{\alpha+}$ between the categories $C(\mathcal{X}), C(\mathcal{X}(U)), C(\mathcal{X}(Z)), C(X_\alpha)$ preserve h-flats. Also the functors $j^*, i_*, i^!, j_{\alpha*}$ preserve h-injective, since their left adjoint functors preserve acyclic complexes.

Derived functors: We have defined the following triangulated functors between the derived categories $D(\mathcal{X})$, $D(\mathcal{X}(U))$, $D(\mathcal{X}(Z))$, $D(X_\alpha)$: $j_!^*, j_!, i_*^*, i_*, \mathbf{R}i_!^!, j_\alpha^*, \mathbf{L}j_{\alpha+}, \mathbf{R}j_{\alpha*}$.

Preserve direct sums: All the above functors except possibly $\mathbf{R}i_!^!$ ($\mathbf{R}j_{\alpha*}$ preserves direct sums since the morphisms $f_{\alpha\beta}$ are quasi-compact and separated [BoVdB], Cor.3.3.4).

Adjunction: $(j_!, j^*), (i^*, i_*), (i_*, \mathbf{R}i_!^!), (j_\alpha^*, \mathbf{R}j_{\alpha*}), (\mathbf{L}j_{\alpha+}, j_\alpha^*)$, are adjoint pairs. This follows (except for the last pair) from the adjunctions in Subsection 2.2 above and the fact that the functors $j^*, i_*, i_!^!, j_{\alpha*}$ preserve h-injectives. For the last pair we need a lemma.

Lemma 3.4. *$(\mathbf{L}j_{\alpha+}, j_\alpha^*)$ is an adjoint pair.*

Proof. Choose $M \in D(X_\alpha)$ and $I \in D(\mathcal{X})$. We need to show that $\mathbf{R}\mathrm{Hom}(\mathbf{L}j_{\alpha+}M, I) = \mathbf{R}\mathrm{Hom}(M, j_\alpha^*I)$. We may assume that M is h-flat and I is a special h-injective (Lemma 3.2). Moreover, we then may assume that $I = j_{\beta*}K, \beta \leq \alpha$ where $K \in C(X_\beta)$ is h-injective. Then $j_\alpha^*I = f_{\beta\alpha*}K$ and so

$$\mathrm{Hom}(M, j_\alpha^*I) = \mathbf{R}\mathrm{Hom}(M, j_\alpha^*I)$$

by Corollary 15.7 in Appendix. Therefore

$$\begin{aligned} \mathbf{R}\mathrm{Hom}(\mathbf{L}j_{\alpha+}M, I) &= \mathrm{Hom}(\mathbf{L}j_{\alpha+}M, I) \\ &= \mathrm{Hom}(j_{\alpha+}M, I) \\ &= \mathrm{Hom}(M, j_\alpha^*I) \\ &= \mathbf{R}\mathrm{Hom}(M, j_\alpha^*I). \end{aligned}$$

□

Definition 3.5. *For $F \in D(\mathcal{X})$ we define the cohomology*

$$H^i(\mathcal{X}, F) := \mathbf{R}^i\mathrm{Hom}(\mathcal{O}_{\mathcal{X}}, F).$$

3.2. Semi-orthogonal decompositions. Recall that functors $j_!$ and i_* identify categories $\mathcal{M}(U)$ and $\mathcal{M}(Z)$ with \mathcal{M}_U and \mathcal{M}_Z respectively. Denote by $D_U(\mathcal{M})$ and $D_Z(\mathcal{M})$ the full subcategories of $D(\mathcal{M})$ consisting of complexes with cohomologies in \mathcal{M}_U and \mathcal{M}_Z respectively.

Lemma 3.6. *The functors $i_* : D(\mathcal{M}(Z)) \rightarrow D(\mathcal{M})$ and $j_! : D(\mathcal{M}(U)) \rightarrow D(\mathcal{M})$ are fully faithful. The essential images of these functors are the full subcategories $D_Z(\mathcal{M})$ and $D_U(\mathcal{M})$ respectively.*

Proof. Given $F \in D_Z(\mathcal{M})$ (resp. $F \in D_U(\mathcal{M})$) the adjunction map $F \rightarrow i_*i^*F$ (resp. $j_!j^*F \rightarrow F$) is a quasiisomorphism. This shows that the functors $i_* : D(\mathcal{M}(Z)) \rightarrow D_Z(\mathcal{M})$ and $j_! : D(\mathcal{M}(U)) \rightarrow D_U(\mathcal{M})$ are essentially surjective. Let us prove that they are fully faithful.

Let $F, G \in D(\mathcal{M}(Z))$ and assume that G is h-injective. Then i_*G is also h-injective and we have

$$\mathbf{R} \operatorname{Hom}(i_*F, i_*G) = \operatorname{Hom}(i_*F, i_*G) = \operatorname{Hom}(i^*i_*F, G) = \mathbf{R} \operatorname{Hom}(F, G).$$

Similarly, let $F, G \in D(\mathcal{M}(U))$ and choose a quasi-isomorphism $j_!G \rightarrow I$, where I is h-injective. Then j^*I is also h-injective and quasi-isomorphic to G . We have

$$\mathbf{R} \operatorname{Hom}(j_!F, j_!G) = \operatorname{Hom}(j_!F, I) = \operatorname{Hom}(F, j^*I) = \mathbf{R} \operatorname{Hom}(F, G).$$

□

We immediately obtain the following corollary

Corollary 3.7. *The categories $D(\mathcal{M}(U))$ and $D(\mathcal{M}(Z))$ are naturally equivalent to $D_U(\mathcal{M})$ and $D_Z(\mathcal{M})$ respectively.*

Corollary 3.8. *Fix $\alpha \in S$. Let $i : \{\alpha\} \hookrightarrow U_\alpha$ and $j : U_\alpha \hookrightarrow S$ be the closed and the open embeddings respectively. Then the functor*

$$j_! \cdot i_* : D(X_\alpha) \rightarrow D(\mathcal{M})$$

is fully faithful. In particular, the derived category $D(X_\alpha)$ is naturally (equivalent to) a full subcategory of $D(\mathcal{M})$.

Proof. Indeed, by Lemma 3.6 above the functors

$$i_* : D(X_\alpha) \rightarrow D(\mathcal{M}(U_\alpha))$$

and

$$j_! : D(\mathcal{M}(U_\alpha)) \rightarrow D(\mathcal{M})$$

are fully faithful. So is their composition. □

Recall the following definitions from [BoKa].

Definition 3.9. *Let \mathcal{A} be a triangulated category, $\mathcal{B} \subset \mathcal{A}$ – a full triangulated subcategory. A right orthogonal to \mathcal{B} in \mathcal{A} is a full subcategory $\mathcal{B}^\perp \subset \mathcal{A}$ consisting of all objects C such that $\operatorname{Hom}(B, C[n]) = 0$ for all $n \in \mathbb{Z}$ and all $B \in \mathcal{B}$.*

Definition 3.10. *Let \mathcal{A} be a triangulated category, $\mathcal{B} \subset \mathcal{A}$ – a full triangulated subcategory. We say that \mathcal{B} is right-admissible if for each $X \in \mathcal{A}$ there exists an exact triangle $B \rightarrow X \rightarrow C$ with $B \in \mathcal{B}$, $C \in \mathcal{B}^\perp$.*

Similarly one defines the left orthogonal to a full subcategory and left admissible subcategories.

Definition 3.11. Let \mathcal{A} be a triangulated category, $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ – two full triangulated subcategories. We say that \mathcal{A} has the semi-orthogonal decomposition $\mathcal{A} = \langle \mathcal{C}, \mathcal{B} \rangle$ if $\mathcal{C} = \mathcal{B}^\perp$ and \mathcal{B} is right-admissible. More generally given full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{A}$ we say that \mathcal{A} has the semi-orthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ if

- 1) \mathcal{A}_1 is right-admissible;
- 2) the right orthogonal \mathcal{A}_1^\perp is the category \mathcal{D} which is the triangulated envelop of the categories $\mathcal{A}_2, \dots, \mathcal{A}_n$;
- 3) there is a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$.

Lemma 3.12. Consider the full subcategories $D_U(\mathcal{M})$ and $D_Z(\mathcal{M})$ of $D(\mathcal{M})$. Then

- i) $D_U(\mathcal{M})^\perp = D_Z(\mathcal{M})$,
- ii) the subcategory $D_U(\mathcal{M}) \subset D(\mathcal{M})$ is right-admissible.

Proof. i). Let $G \in D(\mathcal{M})$. Then $G \in D_U(\mathcal{M})^\perp \simeq (j_! D(\mathcal{M}(U)))^\perp$ iff Gj^* is acyclic, i.e. $G \in D_Z(\mathcal{M})$.

- ii). Given $X \in D(\mathcal{M})$ the required exact triangle is $X_U \rightarrow X \rightarrow X_Z$. □

Corollary 3.13. a) In the notation of Lemma 3.12 we have the semi-orthogonal decomposition $D(\mathcal{M}) = (D_Z(\mathcal{M}), D_U(\mathcal{M}))$.

b) Choose a linear ordering $\alpha_1, \dots, \alpha_n$ of elements of S which is compatible with the given partial order. Using Corollary 3.7 identify each category $D(X_{\alpha_i})$ as a full subcategory of $D(\mathcal{M}) = D(\mathcal{X})$. Then there is the semi-orthogonal decomposition

$$D(\mathcal{X}) = \langle D(X_{\alpha_1}), \dots, D(X_{\alpha_n}) \rangle.$$

Proof. a). This follows directly from the definitions and Lemma 3.12. b) Follows from a) by induction on the cardinality of the poset S . □

4. COMPACT OBJECTS AND PERFECT COMPLEXES ON POSET SCHEMES

Let us first recall the situation with the usual schemes.

Definition 4.1. Let T be a triangulated category.

a) An object $K \in T$ is called compact if the functor $\text{Hom}_T(K, -)$ commutes with direct sums. We denote by $T^c \subset T$ the full triangulated subcategory of compact objects.

b) An object $K \in T^c$ is called a compact generator of T if

$$K^\perp = \{M \in T \mid \text{Hom}(K, M[n]) = 0 \text{ for all } n\} = 0.$$

Definition 4.2. Let X be a scheme. An object $G \in D(X)$ is called perfect if locally it is quasi-isomorphic to a finite complex of free \mathcal{O}_X -modules of finite rank. We denote by $\text{Perf}(X) \subset D(X)$ the full triangulated subcategory of perfect objects.

Theorem 4.3. [BoVdB] *Let X be a scheme. Then*

- a) $\text{Perf}(X) = D(X)^c$,
- b) *the category $D(X)$ has a compact generator.*

As a consequence of this theorem we obtain an equivalence of categories $D(X) \simeq D(A)$ for a DG algebra A . Namely, if $K \in D(X)^c$ is a compact generator and $A = \mathbf{R}\text{Hom}(K, K)$, then the functor

$$\mathbf{R}\text{Hom}(K, -) : D(X) \rightarrow D(A)$$

is an equivalence (see for example [Lu2], Proposition 2.6).

We want to prove analogous results for poset schemes.

Definition 4.4. *Let $\mathcal{X} = \{X_\alpha, f_{\alpha\beta}\}$ be a poset scheme. We call a complex $F = \{F_\alpha\} \in D(\mathcal{X})$ perfect if each $F_\alpha \in D(X_\alpha)$ is such. Denote by $\text{Perf}(\mathcal{X}) \subset D(\mathcal{X})$ the full subcategory of perfect complexes.*

Remark 4.5. *Notice that the functors $j^*, j_!, i^*, i_*, j_\alpha^*, \mathbf{L}j_{\alpha+}$ preserve perfect complexes.*

Proposition 4.6. $D(\mathcal{X})^c = \text{Perf}(\mathcal{X})$.

Proof. Fix a minimal element $\alpha \in S$. Let $U = S - \{\alpha\}$ and denote by $j : U \hookrightarrow S$ and $j_\alpha : \{\alpha\} \hookrightarrow S$ the corresponding open and closed embeddings.

Lemma 4.7. *The functors $j_\alpha^*, j_!$, and $\mathbf{L}j_{\alpha+}$ preserve compact objects.*

Proof. Indeed, their respective right adjoint functors $\mathbf{R}j_{\alpha*}, j^*, j_\alpha^*$ preserve direct sums. \square

By Theorem 4.3 the proposition holds if $|S| = 1$. So by induction we may assume that it holds for X_α and $\mathcal{X}(U)$.

By Lemma 3.6 the functor $j_! : D(\mathcal{X}(U)) \rightarrow D(\mathcal{X})$ is full and faithful with the essential image $D_U(\mathcal{X})$. Let $M \in D_U(\mathcal{X})$ be perfect. Then $j_!^{-1}M \in D(\mathcal{X}(U))$ is also perfect, hence compact by induction. Therefore $M = j_!(j_!^{-1}M) \in D(\mathcal{X})$ is also compact. Vice versa, let $M \in D(\mathcal{X})^c \cap D_U(\mathcal{X})$. Then $M \in D_U(\mathcal{X})^c$ because the inclusion $D_U(\mathcal{X}) \subset D(\mathcal{X})$ preserves direct sums. So $j_!^{-1}(M) \in D(\mathcal{X}(U))^c$. By induction $j_!^{-1}(M)$ is perfect, so M is also perfect. We proved that $D(\mathcal{X})^c \cap D_U(\mathcal{X}) = \text{Perf}(\mathcal{X}) \cap D_U(\mathcal{X})$.

Fix $F \in D(\mathcal{X})^c$. Then $F_\alpha = j_\alpha^*F \in D(X_\alpha)^c$, hence F_α is perfect by induction. Then $\mathbf{L}j_{\alpha+}j_\alpha^*F$ is also compact and perfect. Hence the cone $C(g)$ of the canonical morphism $g : \mathbf{L}j_{\alpha+}j_\alpha^*F \rightarrow F$ is compact. But $C(g) \in D_U(\mathcal{X})$, so $C(g) \in \text{Perf}(\mathcal{X})$. Thus $F \in \text{Perf}(\mathcal{X})$.

Vice versa, let $F \in \text{Perf}(\mathcal{X})$. Then $j^*F \in \text{Perf}(\mathcal{X}(U))$, $j_\alpha^*F \in \text{Perf}(X_\alpha)$. By induction $j^*F \in D(\mathcal{X}(U))^c$ and so $j_!j^*F \in D(\mathcal{X})^c$. Also by induction $j_\alpha^*F \in D(X_\alpha)^c$. Consider the exact triangle

$$j_!j^*F \rightarrow F \rightarrow \mathbf{R}j_{\alpha*}j_\alpha^*F.$$

It suffices to show that $\mathbf{R}j_{\alpha*}j_{\alpha}^*F$ is compact. (Notice that $\mathbf{R}j_{\alpha*}j_{\alpha}^*F$ is perfect because α is a minimal element.) We know that $\mathbf{L}j_{\alpha+}j_{\alpha}^*F$ is perfect and compact. So the cone $C(p)$ of the canonical morphism

$$p : \mathbf{L}j_{\alpha+}j_{\alpha}^*F \rightarrow \mathbf{R}j_{\alpha*}j_{\alpha}^*F$$

is perfect. Also $C(p) \in D_U(\mathcal{X})$. Hence $C(p) \in D(\mathcal{X})^c$ and so also $\mathbf{R}j_{\alpha*}j_{\alpha}^*F$ is compact. \square

4.1. Existence of a compact generator.

Lemma 4.8. *The category $D(\mathcal{X})$ has a compact generator.*

Proof. Choose a compact generator $E_{\alpha} \in D(X_{\alpha})$ for each $\alpha \in S$. Put $E := \bigoplus \mathbf{L}j_{\alpha+}E_{\alpha}$. Then $E \in D(\mathcal{X})^c$, since the functor $\mathbf{L}j_{\alpha+}$ preserves compact objects. For $M \in D(\mathcal{X})$ we have by adjunction

$$\mathrm{Hom}(E, M) = \bigoplus_{\alpha} \mathrm{Hom}(E_{\alpha}, M_{\alpha}).$$

So $\mathrm{Hom}(E[i], M) = 0$ for all i implies that $M = 0$. \square

Definition 4.9. *A compact generator $E \in D(\mathcal{X})$ as constructed in the proof of last lemma will be called special.*

We get the following standard corollary.

Corollary 4.10. *The category $D(\mathcal{X})$ is equivalent to $D(A)$ for a DG algebra A .*

Proof. If E is a compact generator of $D(\mathcal{X})$ and $A = \mathbf{R}\mathrm{Hom}(E, E)$, then the functor

$$\mathbf{R}\mathrm{Hom}(E, -) : D(\mathcal{X}) \rightarrow D(A)$$

is an equivalence of categories. \square

5. SMOOTHNESS OF POSET SCHEMES

In this section we prove the following theorem.

Theorem 5.1. *Let k be a perfect field, S - a (finite) poset and \mathcal{X} a regular S -scheme essentially of finite type. Then the derived category $D(\mathcal{X})$ is smooth.*

Proof. For each $\alpha \in S$ choose a compact generator E_{α} for $D(X_{\alpha})$. Then by (the proof of) Lemma 4.8 the object

$$E := \bigoplus_{\alpha \in S} \mathbf{L}j_{\alpha+}E_{\alpha}$$

is a compact generator for $D(\mathcal{X})$. Put $A := \mathbf{R}\mathrm{Hom}(E, E)$. It suffices to prove that the DG algebra A is smooth.

Choose a minimal element $\delta \in S$, and consider the poset $S' := S - \{\delta\}$. Let $\mathcal{X}' := \mathcal{X} - X_\delta$ be the corresponding S' -scheme.

Since $(\mathbf{L}j_{\alpha+}E_\alpha)|_{X_\delta} = 0$ for each $\alpha \neq \delta$, we may consider

$$E' := \bigoplus_{\alpha \in S'} \mathbf{L}j_{\alpha+}E_\alpha$$

as a compact generator of $D(\mathcal{X}')$. Put $A' := \mathbf{R}\mathrm{Hom}(E', E')$. (The quasi-isomorphism type of A' is independent of where we compute this $\mathbf{R}\mathrm{Hom}$: in $D(\mathcal{X})$ or $D(\mathcal{X}')$.)

By [Lu2], Proposition 3.13 and the induction on $|S|$ we may assume that A' is smooth. Denote

$$A_\delta := \mathbf{R}\mathrm{Hom}(\mathbf{L}j_{\delta+}E_\delta, \mathbf{L}j_{\delta+}E_\delta) \simeq \mathbf{R}\mathrm{Hom}(E_\delta, E_\delta).$$

Then A_δ is also smooth for the same reason. Notice that $\mathbf{R}\mathrm{Hom}(\mathbf{L}j_{\delta+}E_\delta, E') = 0$, hence A is quasi-isomorphic to the triangular DG algebra

$$\begin{pmatrix} A' & 0 \\ {}_{A_\delta}N_{A'} & A_\delta \end{pmatrix},$$

where $N = \mathbf{R}\mathrm{Hom}(E', \mathbf{L}j_{\delta+}E_\delta)$. So by [Lu2], Proposition 3.11 it suffices to show that the DG $A_\delta^{\mathrm{op}} \otimes A'$ -module N is perfect.

Consider the S' -scheme $\mathcal{Y} = \mathcal{X}' \times X_\delta$. That is \mathcal{Y} consists of schemes $X_\alpha \times X_\delta$ for $\alpha \in S'$ and morphisms $f_{\alpha\beta} \times \mathrm{id} : X_\alpha \times X_\delta \rightarrow X_\beta \times X_\delta$. We denote the inclusion $X_\alpha \times X_\delta \rightarrow \mathcal{Y}$ by $j_{(\alpha,\delta)}$.

Let $E_\delta^* := \mathbf{R}\mathrm{Hom}(E_\delta, \mathcal{O}_{X_\delta})$ be the dual compact generator of $D(X_\delta)$. Then $\mathbf{R}\mathrm{Hom}(E_\delta^*, E_\delta^*) \simeq A_\delta^{\mathrm{op}}$ [Lu2], Lemma 3.15. For each $\alpha \in S'$ $E_\alpha \boxtimes E_\delta^*$ is a compact generator of $D(X_\alpha \times X_\delta)$ [BoVdB], Lemma 3.4.1. Thus

$$\tilde{E} := \bigoplus_{\alpha \in S'} \mathbf{L}j_{(\alpha,\delta)+}(E_\alpha \boxtimes E_\delta^*)$$

is a special compact generator for $D(\mathcal{Y})$.

Lemma 5.2. *There is a natural quasi-isomorphism of DG algebras*

$$\mathbf{R}\mathrm{Hom}(\tilde{E}, \tilde{E}) \simeq A_\delta^{\mathrm{op}} \otimes A'.$$

Proof. We have

$$\begin{aligned} \mathbf{R}\mathrm{Hom}(\tilde{E}, \tilde{E}) &\simeq \bigoplus_{\alpha \geq \beta} \mathbf{R}\mathrm{Hom}(\mathbf{L}j_{(\alpha,\delta)+}(E_\alpha \boxtimes E_\delta^*), \mathbf{L}j_{(\beta,\delta)+}(E_\beta \boxtimes E_\delta^*)) \\ &\simeq \bigoplus_{\alpha \geq \beta} \mathbf{R}\mathrm{Hom}(E_\alpha \boxtimes E_\delta^*, \mathbf{L}(f_{\alpha\beta} \times \mathrm{id})^*(E_\beta \boxtimes E_\delta^*)) \\ &\simeq \bigoplus_{\alpha \geq \beta} \mathbf{R}\mathrm{Hom}(E_\alpha \boxtimes E_\delta^*, \mathbf{L}f_{\alpha\beta}^* E_\beta \boxtimes E_\delta^*). \end{aligned}$$

Now by [Lu2], Proposition 6.20

$$\begin{aligned}
 & \mathbf{R} \operatorname{Hom}(E_\alpha \boxtimes E_\delta^*, \mathbf{L} f_{\alpha\beta}^* E_\beta \boxtimes E_\delta^*) \\
 & \simeq \mathbf{R} \operatorname{Hom}(E_\alpha, \mathbf{L} f_{\alpha\beta}^* E_\beta) \otimes \mathbf{R} \operatorname{Hom}(E_\delta^*, E_\delta^*) \\
 & \simeq \mathbf{R} \operatorname{Hom}(E_\alpha, \mathbf{L} f_{\alpha\beta}^* E_\beta) \otimes A_\delta^{\operatorname{op}}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \mathbf{R} \operatorname{Hom}(E', E') & \simeq \bigoplus_{\alpha \geq \beta} \mathbf{R} \operatorname{Hom}(\mathbf{L} j_{\alpha+} E_\alpha, \mathbf{L} j_{\beta+} E_\beta) \\
 & \simeq \bigoplus_{\alpha \geq \beta} \mathbf{R} \operatorname{Hom}(E_\alpha, \mathbf{L} f_{\alpha\beta}^* E_\beta).
 \end{aligned}$$

This proves the lemma. \square

It follows that the functor

$$\Psi_{\tilde{E}}(-) := \mathbf{R} \operatorname{Hom}(\tilde{E}, -) : D(\mathcal{Y}) \rightarrow D(A_\delta^{\operatorname{op}} \otimes A')$$

is an equivalence of categories.

For each $\alpha \in S'$, such that $\alpha > \delta$ denote by $\Gamma(\alpha, \delta) \subset X_\alpha \times X_\delta$ the graph of the map $f_{\alpha,\delta} : X_\alpha \rightarrow X_\delta$. Define the coherent sheaf F on \mathcal{Y} as follows. For $\alpha \in S'$ such that $\alpha > \delta$ put $F_\alpha := \mathcal{O}_{\Gamma(\alpha\delta)} \in \operatorname{coh}(X_\alpha \times X_\delta)$. If $\delta \not\leq \alpha$, then put $F_\alpha = 0$. The structure morphism $\phi_{\alpha\beta} : f_{\alpha\beta}^* F_\beta \rightarrow F_\alpha$ is the canonical isomorphism.

Lemma 5.3. *We have $\Psi_{\tilde{E}}(F) \simeq N$.*

Proof. By definition

$$\begin{aligned}
 N &= \mathbf{R} \operatorname{Hom}_{\mathcal{X}}(E', \mathbf{L} j_{\delta+} E_\delta) \\
 &= \bigoplus_{\alpha \in S'} \mathbf{R} \operatorname{Hom}_{\mathcal{X}}(\mathbf{L} j_{\alpha+} E_\alpha, \mathbf{L} j_{\delta+} E_\delta) \\
 &= \bigoplus_{\alpha \in S'} \mathbf{R} \operatorname{Hom}_{X_\alpha}(E_\alpha, \mathbf{L} f_{\alpha\delta}^* E_\delta)
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 \mathbf{R} \operatorname{Hom}_{\mathcal{Y}}(\tilde{E}, F) &= \bigoplus_{\alpha \in S'} \mathbf{R} \operatorname{Hom}_{\mathcal{Y}}(\mathbf{L} j_{(\alpha,\delta)+}(E_\alpha \boxtimes E_\delta^*), F) \\
 &= \bigoplus_{\alpha \in S'} \mathbf{R} \operatorname{Hom}_{X_\alpha \times X_\delta}(E_\alpha \boxtimes E_\delta^*, \mathcal{O}_{\Gamma(\alpha\delta)})
 \end{aligned}$$

Let us analyze one summand in the last sum. Denote by $E_\alpha \xleftarrow{p_\alpha} E_\alpha \times E_\delta \xrightarrow{p_\delta} E_\delta$ and by $\gamma : \Gamma(\alpha\delta) \rightarrow X_\delta$ the obvious projections.

$$\begin{aligned}
 & \mathbf{R} \operatorname{Hom}(E_\alpha \boxtimes E_\delta^*, \mathcal{O}_{\Gamma(\alpha\delta)}) \\
 &= \mathbf{R} \operatorname{Hom}(p_\alpha^* E_\alpha \otimes p_\delta^* \mathbf{R} \operatorname{Hom}(E_\delta, \mathcal{O}_{X_\delta}), \mathcal{O}_{\Gamma(\alpha\delta)}) \\
 &= \mathbf{R} \operatorname{Hom}(p_\alpha^* E_\alpha, \mathbf{R} \operatorname{Hom}(p_\delta^* \mathbf{R} \operatorname{Hom}(E_\delta, \mathcal{O}_{X_\delta}), \mathcal{O}_{\Gamma(\alpha\delta)})) \\
 &= \mathbf{R} \operatorname{Hom}(p_\alpha^* E_\alpha, \mathbf{R} \operatorname{Hom}_{\Gamma(\alpha\delta)}(\mathbf{L} \gamma_\delta^* \mathbf{R} \operatorname{Hom}(E_\delta, \mathcal{O}_{X_\delta}), \mathcal{O}_{\Gamma(\alpha\delta)})) \\
 &= \mathbf{R} \operatorname{Hom}(p_\alpha^* E_\alpha, \mathbf{R} \operatorname{Hom}_{\Gamma(\alpha\delta)}(\mathbf{R} \operatorname{Hom}_{\Gamma(\alpha\delta)}(\mathbf{L} \gamma^* E_\delta, \mathcal{O}_{\Gamma(\alpha\delta)}), \mathcal{O}_{\Gamma(\alpha\delta)})) \\
 &= \mathbf{R} \operatorname{Hom}(p_\alpha^* E_\alpha, \mathbf{L} \gamma^* E_\delta) \\
 &= \mathbf{R} \operatorname{Hom}(E_\alpha, \mathbf{L} f_{\alpha\delta}^* E_\delta).
 \end{aligned}$$

This proves the lemma. \square

Since the poset scheme \mathcal{Y} is regular the object $F \in D(\mathcal{Y})$ is compact by Proposition 4.6. Hence $N \simeq \Psi_{\tilde{E}}(F) \in D(A_{\delta}^{\text{op}} \otimes A')$ is also compact, i.e. is perfect. This proves Theorem 5.1. \square

6. DIRECT AND INVERSE IMAGE FUNCTORS FOR MORPHISMS OF POSET SCHEMES

Let S, S' be posets and $\tau : S \rightarrow S'$ be an order preserving map. Let $\mathcal{X} = \{X_{\alpha}, f_{\alpha\beta}\}$ (resp. $\mathcal{X}' = \{X'_{\alpha'}, f_{\alpha'\beta'}\}$) be an S -scheme (resp. an S' -scheme).

Definition 6.1. A τ -morphism $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{X}'$ is a collection of morphisms $\{\mathcal{F}_{\alpha} : X_{\alpha} \rightarrow X'_{\tau(\alpha)}\}_{\alpha \in S}$ such that for each $\alpha \geq \beta$ the following diagram commutes

$$\begin{array}{ccc} X_{\alpha} & \xrightarrow{f_{\alpha\beta}} & X_{\beta} \\ \downarrow \mathcal{F}_{\alpha} & & \downarrow \mathcal{F}_{\beta} \\ X'_{\tau(\alpha)} & \xrightarrow{f'_{\tau(\alpha)\tau(\beta)}} & X'_{\tau(\beta)} \end{array}$$

Let $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{X}'$ be a τ -morphism and $G \in Qcoh_{\mathcal{X}'}$. We define $\mathcal{F}^*G \in Qcoh_{\mathcal{X}}$ as follows. For $\alpha \in S$ put $(\mathcal{F}^*G)_{\alpha} = \mathcal{F}_{\alpha}^*G_{\tau(\alpha)}$ and define the structure morphism $\phi_{\alpha\beta} : f_{\alpha\beta}^* \mathcal{F}_{\beta}^* G_{\tau(\beta)} \rightarrow \mathcal{F}_{\alpha}^* G_{\tau(\alpha)}$ as $\mathcal{F}_{\alpha}^* \phi'_{\tau(\alpha)\tau(\beta)}$, where ϕ' is the structure morphism for G . This defines a functor $\mathcal{F}^* : Qcoh_{\mathcal{X}'} \rightarrow Qcoh_{\mathcal{X}}$. We also consider its left derived functor $\mathbf{L}\mathcal{F}^* : D(\mathcal{X}') \rightarrow D(\mathcal{X})$ which is defined using the h-flats.

Notice that the functor \mathcal{F}^* preserves h-flats.

Example 6.2. We have $\mathcal{F}^* \mathcal{O}_{\mathcal{X}'} = \mathbf{L}\mathcal{F}^* \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}}$. Hence, for any $G \in D(\mathcal{X}')$ we obtain the map $\mathcal{F}^* : H^{\bullet}(\mathcal{X}', G) \rightarrow H^{\bullet}(\mathcal{X}, \mathbf{L}\mathcal{F}^*G)$; in particular we get the map $\mathcal{F}^* : H^{\bullet}(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \rightarrow H^{\bullet}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Also the usual morphism $\mathcal{F}^* \Omega_{\mathcal{X}'}^i \rightarrow \Omega_{\mathcal{X}}^i$ induces the map

$$H^{\bullet}(\mathcal{X}', \Omega_{\mathcal{X}'}^i) \rightarrow H^{\bullet}(\mathcal{X}, \Omega_{\mathcal{X}}^i).$$

Given another morphism of poset schemes $\mathcal{F}' : \mathcal{X}' \rightarrow \mathcal{X}''$ there are natural isomorphisms of functors $\mathcal{F}^* \mathcal{F}'^* \simeq (\mathcal{F}' \mathcal{F})^*$. Since the functor \mathcal{F}'^* preserves h-flats we also have an isomorphism $\mathbf{L}\mathcal{F}^* \cdot \mathbf{L}\mathcal{F}'^* \simeq \mathbf{L}(\mathcal{F}' \mathcal{F})^*$.

The functor \mathcal{F}^* has the right adjoint functor \mathcal{F}_* which we now describe.

We will use Remark 2.4

For $\alpha' \in S'$ we put $\tau^{-1}(\geq \alpha') := \{\gamma \in S \mid \tau(\gamma) \geq \alpha'\}$. Fix $F \in Qcoh_{\mathcal{X}}$. If $\gamma \in \tau^{-1}(\geq \alpha')$, then $f'_{\tau(\gamma)\alpha'}(\mathcal{F}_{\gamma}^* F_{\gamma}) \in Qcoh_{X'_{\alpha'}}$. If $\delta \geq \gamma$, then the structure morphism $\psi_{\delta\gamma} : F_{\gamma} \rightarrow f_{\delta\gamma}^* F_{\delta}$ induces the morphism

$$f'_{\tau(\gamma)\alpha'}(\mathcal{F}_{\gamma}^* F_{\gamma}) \rightarrow f'_{\tau(\delta)\alpha'}(\mathcal{F}_{\delta}^* F_{\delta}).$$

We define

$$(\mathcal{F}_* F)_{\alpha'} = \varprojlim_{\gamma \in \tau^{-1}(\geq \alpha')} f'_{\tau(\gamma)\alpha'}(\mathcal{F}_{\gamma}^* F_{\gamma}).$$

If $\alpha' \geq \beta'$ there is a natural morphism $\psi'_{\alpha'\beta'} : f'_{\alpha'\beta'}(\mathcal{F}_*F)_{\alpha'} \rightarrow (\mathcal{F}_*F)_{\beta'}$. Thus $\mathcal{F}_*F \in Qcoh\mathcal{X}'$ and we get a functor $\mathcal{F}_* : Qcoh\mathcal{X} \rightarrow Qcoh\mathcal{X}'$. We define its right derived functor $\mathbf{R}\mathcal{F}_* : D(\mathcal{X}) \rightarrow D(\mathcal{X}')$ using the h-injectives. The pairs of functors $(\mathcal{F}^*, \mathcal{F}_*)$ and $(\mathbf{L}\mathcal{F}^*, \mathbf{R}\mathcal{F}_*)$ are adjoint.

Given another morphism of poset schemes $\mathcal{F}' : \mathcal{X}' \rightarrow \mathcal{X}''$ there are natural isomorphisms of functors $\mathcal{F}'_*\mathcal{F}_* \simeq (\mathcal{F}'\mathcal{F})_*$. Although the functor \mathcal{F}_* may not preserve h-injectives we still have a natural isomorphism of functors $\mathbf{R}\mathcal{F}'_* \cdot \mathbf{R}\mathcal{F}_* \simeq \mathbf{R}(\mathcal{F}'\mathcal{F})_*$ (this follows by adjunction from the isomorphism $\mathbf{L}\mathcal{F}^* \cdot \mathbf{L}\mathcal{F}'^* \simeq \mathbf{L}(\mathcal{F}'\mathcal{F})^*$).

The direct image functor may be computed fiberwise in case τ is the projection of a product poset on one of the factors. Namely we have the following lemma.

Lemma 6.3. *Assume that T is a poset, $S = S' \times T$ is the product poset and $\tau : S \rightarrow S'$ is the projection. Then in the above notation for any $\alpha' \in S'$ we have*

$$(\mathcal{F}_*F)_{\alpha'} = \varprojlim_{\gamma \in \tau^{-1}(\alpha')} \mathcal{F}_{\gamma*}F_{\gamma}.$$

Proof. This is clear. □

Example 6.4. *Let S' consist of a single element α' and $X_{\alpha'} = pt$. Then for $F \in D(\mathcal{X})$*

$$\mathbf{R}^i\mathcal{F}_*F = H^i(\mathcal{X}, F).$$

7. CATEGORICAL RESOLUTIONS BY SMOOTH POSET SCHEMES

Let S be a (finite) poset and let \mathcal{X} be a smooth $(S-)$ poset scheme (so that the category $D(\mathcal{X})$ is smooth by Theorem 5.1). Let Y be a scheme (which can be considered as a poset scheme) and $\pi : \mathcal{X} \rightarrow Y$ be a morphism of poset schemes (i.e. a τ -morphism for $\tau : S \rightarrow pt$, in the terminology of the previous section).

Definition 7.1. *We call the morphism π a categorical resolution of Y if the functor $\mathbf{L}\pi^* : D(Y) \rightarrow D(\mathcal{X})$ is a categorical resolution, i.e. its restriction $\mathbf{L}\pi^* : \text{Perf}(Y) \rightarrow \text{Perf}(\mathcal{X})$ is full and faithful.*

We can localize the morphism π over Y in the obvious way. Namely, given an open subset $W \subset Y$ we denote by \mathcal{X}_W the poset scheme which is the universe image of W under π . Let $\pi_W : \mathcal{X}_W \rightarrow W$ be the corresponding morphism. If W is affine $W = \text{Spec}B$, then the B -module $\mathbf{R}^j(\pi_W)_*\mathcal{O}_{\mathcal{X}_W}$ is isomorphic to $H^j(\mathcal{X}_W, \mathcal{O}_{\mathcal{X}_W})$.

Proposition 7.2. *Let \mathcal{X} be a smooth poset scheme, Y be a scheme and $\pi : \mathcal{X} \rightarrow Y$ be a morphism. The following statements are equivalent.*

- 1) π is a categorical resolution;
- 2) the adjunction morphism $\mathcal{O}_Y \rightarrow \mathbf{R}\pi_*\mathcal{O}_{\mathcal{X}}$ is a quasi-isomorphism;

3) for each affine open set $W \subset Y$ the map $H^0(W, \mathcal{O}_W) \rightarrow H^0(\mathcal{X}_W, \mathcal{O}_{\mathcal{X}_W})$ is an isomorphism and $H^j(\mathcal{X}_W, \mathcal{O}_{\mathcal{X}_W}) = 0$ for $j > 0$.

Proof. The equivalence $2) \Leftrightarrow 3)$ is clear. It remains to prove the equivalence $1) \Leftrightarrow 2)$.

Lemma 7.3. *Let \mathcal{C}, \mathcal{D} be categories, $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor and $G : \mathcal{D} \rightarrow \mathcal{C}$ its right adjoint functor. Fix an object $B \in \mathcal{C}$. Then the following assertions are equivalent*

a) *For any object $A \in \mathcal{C}$ the map $F : \text{Hom}(A, B) \rightarrow \text{Hom}(F(A), F(B))$ is an isomorphism;*

b) *The adjunction morphism $I_B : B \rightarrow GF(B)$ is an isomorphism.*

Proof. The composition of the map $\text{Hom}(A, B) \xrightarrow{F} \text{Hom}(F(A), F(B))$ with the canonical isomorphism $\text{Hom}(F(A), F(B)) \simeq \text{Hom}(A, GF(B))$ is equal to the map $(I_B)_* : \text{Hom}(A, B) \rightarrow \text{Hom}(A, GF(B))$. \square

Now we can prove the equivalence $1) \Leftrightarrow 2)$.

Since the functor $\mathbf{L}\pi^* : D(Y) \rightarrow D(\mathcal{X})$ preserves direct sums and perfect complexes (i.e. compact objects) it is easy to see that it is full and faithful if and only if its restriction to the subcategory $\text{Perf}(Y)$ is such. (Use the fact that $D(Y)$ is the smallest triangulated subcategory of $D(Y)$ which contains $\text{Perf}(Y)$ and is closed under direct sums.) Hence by Lemma 7.3 the functor $\mathbf{L}\pi^* : \text{Perf}(Y) \rightarrow \text{Perf}(\mathcal{X})$ is full and faithful if and only if for every $K \in \text{Perf}(Y)$ the adjunction map $K \rightarrow \mathbf{R}\pi_* \mathbf{L}\pi^* K$ is an isomorphism. But the last assertion is local on Y , and locally K is isomorphic to a finite direct sum of shifted copies of the structure sheaf. \square

We give examples of categorical resolutions by smooth poset schemes in Section 13 below.

8. HOW TO COMPUTE IN $D(\mathcal{X})$

The restriction of an h-injective object $I \in D(\mathcal{X})$ to $X_\alpha \in \mathcal{X}$ may not be h-injective.

Example 8.1. $\mathcal{X} = \{\text{pt} \rightarrow \mathbb{A}^1\}$ and $I = j_*(k)$, where j is the inclusion of the point pt in \mathcal{X} . Then the object $I \in Q\text{coh}\mathcal{X}$ is injective, hence h-injective as an object in $D(\mathcal{X})$, but its restriction to \mathbb{A}^1 is not.

Nevertheless if $I \in D(\mathcal{X})$ is h-injective, then the object $I_\alpha \in D(X_\alpha)$ can be used to compute $\mathbf{R}\text{Hom}(M, -)$, if $M \in D(X_\alpha)$ is h-flat.

Lemma 8.2. *Let $I \in D(\mathcal{X})$ be h-injective. Fix $\alpha \in S$ and let $M \in D(X_\alpha)$ be h-flat. Then the complex $\text{Hom}(M, I_\alpha)$ is quasi-isomorphic to $\mathbf{R}\text{Hom}(M, I_\alpha)$.*

Proof. A proof of this lemma is contained in the proof of Lemma 3.4 above. \square

Lemma 8.3. (a) Fix $\alpha \in S$ and let $F \in D(\mathcal{X})$ be such that $F = j_{\alpha+}F_\alpha$ for an h -flat $F_\alpha \in D(X_\alpha)$. Then for any $G \in D(\mathcal{X})$ we have

$$\mathrm{Hom}_{D(\mathcal{X})}(F, G) = \mathrm{Hom}_{D(X_\alpha)}(F_\alpha, G_\alpha).$$

(b) Suppose that $\alpha \in S$ is the unique minimal element of S , i.e. $S = U_\alpha$ (Subsection 2.1). Then for any $G \in D(\mathcal{X})$

$$H^\bullet(\mathcal{X}, G) = H^\bullet(X_\alpha, G_\alpha).$$

Proof. By Lemma 3.4 the functors $(\mathbf{L}j_{\alpha+}, j_{\alpha+}^*)$ are adjoint, which implies (a). Now (b) follows because $\mathcal{O}_\mathcal{X} = j_{\alpha+}\mathcal{O}_{X_\alpha}$. \square

The next proposition generalizes the last lemma.

Proposition 8.4. Suppose that a complex $F \in C(\mathcal{X})$ has a resolution (in $C(\mathcal{X})$)

$$(8.1) \quad 0 \rightarrow K_n \rightarrow \dots \rightarrow K_1 \rightarrow K_0 \rightarrow F \rightarrow 0$$

where for each i , $K_i = \bigoplus_\alpha j_{\alpha+}M_\alpha^i$ with $M_\alpha^i \in C(X_\alpha)$ being h -flat. Let $I \in C(\mathcal{X})$ be such that for each $\alpha \in S$ and each i , $\mathrm{Hom}(M_\alpha^i, I_\alpha) = \mathbf{R}\mathrm{Hom}(M_\alpha^i, I_\alpha)$ (for example I is h -injective as in Lemma 8.2). Then the complex $\mathbf{R}\mathrm{Hom}(F, I)$ is quasi-isomorphic to the total complex of the double complex

$$(8.2) \quad 0 \rightarrow \mathrm{Hom}(K_0, I) \rightarrow \mathrm{Hom}(K_1, I) \rightarrow \dots \rightarrow \mathrm{Hom}(K_n, I) \rightarrow 0.$$

Moreover, for each i

$$\mathrm{Hom}(K_i, I) = \bigoplus_\alpha \mathrm{Hom}(\mathbf{L}j_{\alpha+}M_\alpha^i, I) = \bigoplus_\alpha \mathrm{Hom}(M_\alpha^i, I_\alpha) = \bigoplus_\alpha \mathbf{R}\mathrm{Hom}(M_\alpha^i, I_\alpha).$$

Hence in particular we obtain a spectral sequence which converges to $\mathrm{Ext}(F, I)$ with the E_1 -term being the sum of groups $\mathrm{Ext}(M_\alpha^i, I_\alpha)$.

Proof. This follows from Lemma 3.4 and Lemma 8.2. \square

The following example will be of primary interest to us.

Example 8.5. In case $F = \mathcal{O}_\mathcal{X}$ one can take a resolution 8.1 with $K_i = \bigoplus_\alpha j_{\alpha+}\mathcal{O}_{X_\alpha}$, i.e. $M_\alpha^i = \mathcal{O}_{X_\alpha}$. (The same index α may appear in different K_i 's and it may also appear more than once in a given K_i .) Given $G \in D(\mathcal{X})$ choose its h -injective replacement I . Then the double complex 8.2 consists of sums of spaces $\Gamma(X_\alpha, I_\alpha)$ and the E_1 -term is the sum of groups $H^\bullet(X_\alpha, G_\alpha)$. The differential d_1 between the cohomology groups is simply the sum of the maps induced by the structure morphisms $\phi_{\alpha\beta} : f_{\alpha\beta}^*G_\beta \rightarrow G_\alpha$. In particular d_1 preserves the degree of the cohomology groups $H^\bullet(X_\alpha, G_\alpha)$.

In case the complex $G \in D(\mathcal{X})$ is bounded below we can use instead of an h -injective I the canonical Godement resolution $G \rightarrow \mathcal{C}^\bullet(G)$, such that for each α the complex $\mathcal{C}^\bullet(G)$ consists of flabby sheaves. Notice that the complex $\mathcal{C}^\bullet(G)$ consists of $\mathcal{O}_\mathcal{X}$ -modules which are no longer quasi-coherent (see Section 9 below).

Definition 8.6. We call any spectral sequence converging to $H^\bullet(\mathcal{X}, G)$ as in the above example a standard one. (It is not unique because one can choose different resolutions 8.1 of $\mathcal{O}_\mathcal{X}$.)

Example 8.7. Assume that a poset S consists of 4 elements $\{\alpha, \beta_1, \beta_2, \beta_3\}$ where $\alpha \geq \beta_i$ for all i and no other relations. Therefore we have 4 irreducible open subsets $U_\alpha, U_{\beta_i} \subset S$. If \mathcal{X} is an S -scheme one can take for example the following resolution 8.1 of the structure sheaf $\mathcal{O}_\mathcal{X}$:

$$0 \rightarrow K_1 \rightarrow K_0 \rightarrow \mathcal{O}_\mathcal{X} \rightarrow 0,$$

where $K_0 = \oplus_i j_{\beta_i+} \mathcal{O}_{X_{\beta_i}}$ and $K_1 = (j_{\alpha+} \mathcal{O}_{X_\alpha})^{\oplus 2}$. This gives a standard spectral sequence converging to $H^\bullet(\mathcal{X}, \mathcal{O}_\mathcal{X})$ with the E_1 -complex

$$\bigoplus_i H^\bullet(X_{\beta_i}, \mathcal{O}_{X_{\beta_i}}) \rightarrow H^\bullet(X_\alpha, \mathcal{O}_{X_\alpha})^{\oplus 2}.$$

Part 2. Poset schemes and Du Bois singularities

9. OTHER VARIANTS OF POSET RINGED SPACES

Besides poset schemes and quasi-coherent sheaves on them we can consider "poset" versions of other usual structures. We give some examples which will be used later. Let \mathcal{X} be a poset scheme.

1) One may define an abelian category $\text{Mod } \mathcal{O}_\mathcal{X}$ just as we defined $Qcoh \mathcal{X}$ by requiring the sheaves F_α to be arbitrary \mathcal{O}_{X_α} -modules and not necessarily quasi-coherent ones. Moreover we may consider the abelian category $\text{Sh}(\mathcal{X})$ of sheaves of abelian groups on \mathcal{X} . (That is we consider each X_α as a ringed space with the structure sheaf \mathbb{Z}_{X_α} , so that the gluing is by maps $\phi'_{\alpha\beta} : f_{\alpha\beta}^{-1} F_\beta \rightarrow F_\alpha$.) Because of the natural morphism $f_{\alpha\beta}^{-1} F_\beta \rightarrow f_{\alpha\beta}^* F_\beta$ each object in $\text{Mod } \mathcal{X}$ defines an object of $\text{Sh}(\mathcal{X})$.

2) Denote by \mathcal{X}^{et} the same diagram of schemes where we consider each X_α with the étale topology. Let $\text{Sh}(\mathcal{X}^{\text{et}})$ denote the abelian category of sheaves of abelian groups on \mathcal{X} . For a prime number l and $n \geq 1$ let $\text{Sh}_{l^n}(\mathcal{X}^{\text{et}}) \subset \text{Sh}(\mathcal{X}^{\text{et}})$ be the full subcategory of \mathbb{Z}/l^n -modules.

3) If \mathcal{X} is a complex poset scheme of finite type we may consider the corresponding poset analytic space \mathcal{X}^{an} . It comes with the structure sheaf $\mathcal{O}_{\mathcal{X}^{\text{an}}}$. (We will be interested in \mathcal{X}^{an} only for projective \mathcal{X} .) Again we denote by $\text{Sh}(\mathcal{X}^{\text{an}})$ the abelian category of

sheaves of abelian groups on \mathcal{X}^{an} . As in the algebraic case, a sheaf of $\mathcal{O}_{\mathcal{X}^{\text{an}}}$ -modules may be considered as an element of $\text{Sh}(\mathcal{X}^{\text{an}})$. In particular the analytic deRham complex $\Omega_{\mathcal{X}^{\text{an}}}^\bullet$ is a complex in $\text{Sh}(\mathcal{X}^{\text{an}})$ which is a resolution of the constant sheaf $\mathbb{C}_{\mathcal{X}^{\text{an}}}$.

All the functors defined in Section 2.1 for quasi-coherent sheaves exist also in the categories described in 1),2),3) above. They have all the properties listed in Subsection 2.2.

Lemma 9.1. *There are enough injectives in all the above categories $\text{Mod } \mathcal{O}_{\mathcal{X}}$, $\text{Sh}(\mathcal{X})$, $\text{Sh}(\mathcal{X}^{\text{et}})$, $\text{Sh}_{I^n}(\mathcal{X}^{\text{et}})$, $\text{Sh}(\mathcal{X}^{\text{an}})$, etc.*

Proof. The proof is essentially the same as the one of Proposition 2.8. \square

Definition 9.2. *Using the above lemma we may define for each bounded below complex L of sheaves in $\text{Sh}(\mathcal{X}^?)$ its cohomology*

$$H^\bullet(\mathcal{X}^?, L) = \text{Ext}^\bullet(\mathbb{Z}_{\mathcal{X}^?}, L)$$

Let L is a bounded above complex of sheaves in one of the categories in Lemma 9.1. There is a spectral sequence converging to $H^\bullet(\mathcal{X}^?, L)$ defined similarly to Example 8.5. Namely, choose a resolution

$$0 \rightarrow K_n \rightarrow \dots \rightarrow K_0 \rightarrow \mathbb{Z}_{\mathcal{X}^?} \rightarrow 0$$

where each K_i is a direct sum of objects $j_{\alpha+}\mathbb{Z}_{X_\alpha}$, which are extensions by zero from irreducible open subsets U_α of the constant sheaf \mathbb{Z} . Choose also an injective resolution $L \rightarrow I$. Then exactly as in Section 8 we get a spectral sequence which converges to $H^\bullet(\mathcal{X}, L)$. The E_0 -term consists of sums of spaces

$$\Gamma(X_\alpha, I_\alpha) = \text{Hom}(j_{\alpha+}\mathbb{Z}_{X_\alpha}, I_\alpha)$$

and the E_1 -term is the sum of cohomologies $H^\bullet(X_\alpha, L_\alpha)$.

Notice that instead of an injective resolution $L \rightarrow I$ we could use the canonical flabby Godement resolution $L \rightarrow G(L)$. (Since the Godement resolution of usual sheaves is functorial it extends to poset sheaves in $\text{Sh}(\mathcal{X}^?)$.)

Definition 9.3. *As in the case of quasi-coherent sheaves (Definition 8.6) we call the above spectral sequence converging to $H^\bullet(\mathcal{X}^?, L)$ a standard one.*

Remark 9.4. *Assume that L is a bounded below complex in $Q\text{coh}\mathcal{X}$. By comparing the corresponding standard spectral sequences we conclude that the cohomology of L is the same whether we consider L as a complex over $Q\text{coh}\mathcal{X}$ or over $\text{Sh}(\mathcal{X})$.*

9.1. Poset GAGA. Let X be a complex projective variety, X^{an} - the corresponding analytic space and $\iota : X^{\text{an}} \rightarrow X$ the canonical morphism of locally ringed spaces. For an \mathcal{O}_X -module F we denote by $F^{\text{an}} = \iota^* F$ its analytization. By adjunction we obtain the canonical morphism of sheaves $a_F : F \rightarrow \iota_* F^{\text{an}}$. Let Y be another complex projective variety and $f : X \rightarrow Y$ be a morphism. The adjunction morphism a_F induces a morphism of sheaves $\theta_F : (f_* F)^{\text{an}} \rightarrow f_*^{\text{an}} F^{\text{an}}$. If F is coherent then it is known by [SGAI], Expose XII, Th. 4.2 (which is an extension of GAGA) that this morphism θ_F induces a quasi-isomorphism $(\mathbf{R}f_* F)^{\text{an}} \rightarrow \mathbf{R}f_*^{\text{an}} F^{\text{an}}$. In particular $H(X, F) = H(X^{\text{an}}, F^{\text{an}})$ for a coherent sheaf F .

Let S be a poset, let \mathcal{X} be a complex projective S -scheme, and $F \in \text{Mod } \mathcal{O}_{\mathcal{X}}$. Again we denote by F^{an} - the analytization of F - the corresponding analytic sheaf on the poset analytic space \mathcal{X}^{an} . The poset analogue of the adjunction map a_F above induces a morphism of the standard spectral sequences for $H^\bullet(\mathcal{X}, F)$ and $H^\bullet(\mathcal{X}^{\text{an}}, F^{\text{an}})$. If $F \in \text{coh } \mathcal{X}$ then it follows from the above cited result in [SGAI] that the induced morphism of E_1 -terms is an isomorphism. In particular for a coherent F we have $H^\bullet(\mathcal{X}, F) = H^\bullet(\mathcal{X}^{\text{an}}, F^{\text{an}})$. Moreover for $F \in \text{coh } \mathcal{X}$ the standard spectral sequence for $H^\bullet(\mathcal{X}, F)$ degenerates at E_r for $r \geq 2$ if and only if the standard spectral sequence for $H^\bullet(\mathcal{X}^{\text{an}}, F^{\text{an}})$ degenerates at E_r . All the above holds also for bounded below complexes of coherent sheaves on \mathcal{X} .

Let S' be another poset and $\tau : S \rightarrow S'$ - a map of posets. Let \mathcal{X}' be a complex projective S' -scheme and $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{X}'$ - a τ -morphism (Definition 6.1). Then for $F \in \text{coh } \mathcal{X}$ there is a natural quasi-isomorphism of complexes of sheaves on \mathcal{X}'^{an}

$$(\mathbf{R}\mathcal{F}_* F)^{\text{an}} \xrightarrow{\sim} \mathbf{R}\mathcal{F}_*^{\text{an}} F^{\text{an}}.$$

In particular, for the deRham complex $\Omega_{\mathcal{X}}^\bullet$ we have

$$(\mathbf{R}\mathcal{F}_* \Omega_{\mathcal{X}}^\bullet)^{\text{an}} \simeq \mathbf{R}\mathcal{F}_*^{\text{an}} \Omega_{\mathcal{X}^{\text{an}}}^\bullet.$$

10. DEGENERATION OF THE STANDARD SPECTRAL SEQUENCE FOR $H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C})$ WHEN \mathcal{X} IS A SMOOTH PROJECTIVE POSET SCHEME

Theorem 10.1. *Let \mathcal{X} be a smooth complex projective poset scheme. Then the standard spectral sequence converging to $H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C}) = H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C}_{\mathcal{X}^{\text{an}}})$ degenerates at E_2 ($d_2 = d_3 = \dots = 0$). That is the cohomology $H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C})$ is isomorphic to the cohomology of the complex*

$$(10.1) \quad E_1 = \dots \rightarrow \oplus H^\bullet(X_\beta^{\text{an}}, \mathbb{C}) \xrightarrow{\oplus f_{\alpha\beta}^*} \oplus H^\bullet(X_\alpha^{\text{an}}, \mathbb{C}) \rightarrow \dots$$

Proof. We use Weil conjectures (Deligne's theorem) [De] to prove this. We follow the strategy of [BBD], Ch.6 using canonical Godement flabby resolutions as in [FK], Ch.1, Sect.11,12.

The argument has three steps: first we pass from the analytic topology to the etale one, then pass to a poset scheme over a finite field, and finally we use purity of the Frobenius endomorphism on the etale l -adic cohomology of a smooth projective scheme.

Step 1. Choose a prime number l . Since the fields $\bar{\mathbb{Q}}_l$ and \mathbb{C} are isomorphic, it suffices to prove the degeneration of the analogous spectral sequence for the cohomology groups $H^\bullet(\mathcal{X}^{\text{an}}, \bar{\mathbb{Q}}_l)$.

Let Y be a complex scheme. We have the natural morphism of topoi $\iota : Y^{\text{an}} \rightarrow Y^{\text{et}}$. This morphism induces the inverse image functor between the corresponding categories of abelian sheaves $\iota^* : \text{Sh}(Y^{\text{et}}) \rightarrow \text{Sh}(Y^{\text{an}})$. It has the following properties [FK], Ch.1, Prop.11.4.

- Given a morphism of schemes $f : X \rightarrow Y$ there is a natural isomorphism of functors $f^{\text{an}*} \cdot \iota_Y^* = \iota_X^* \cdot f^{\text{et}*}$. In particular, ι^* is an exact functor.
- For any point $y \in Y^{\text{an}}$ and any $F \in \text{Sh}(Y^{\text{et}})$ the stalks F_y and $(\iota^* F)_y$ are naturally isomorphic.
- For a finite ring R we have $\iota^*(R_{Y^{\text{et}}}) = R_{Y^{\text{an}}}$ and it induces an isomorphism $H^\bullet(Y^{\text{an}}, R) = H^\bullet(Y^{\text{et}}, R)$.

Recall that the cohomology groups $H^\bullet(Y^{\text{et}}, \bar{\mathbb{Q}}_l)$ are defined as

$$H^\bullet(Y^{\text{et}}, \bar{\mathbb{Q}}_l) := (\lim_{\leftarrow} H^\bullet(Y^{\text{et}}, \mathbb{Z}/l^n)) \otimes_{\mathbb{Z}_l} \bar{\mathbb{Q}}_l$$

It is known that the morphism ι induces an isomorphism $\iota^* : H^\bullet(Y^{\text{et}}, \bar{\mathbb{Q}}_l) \rightarrow H^\bullet(Y^{\text{an}}, \bar{\mathbb{Q}}_l)$. We want to extend this result to poset schemes.

Namely, let \mathcal{X}^{et} denote the poset scheme \mathcal{X} considered in the etale topology. Similarly to the analytic case we define the cohomology groups $H^\bullet(\mathcal{X}^{\text{et}}, \mathbb{Z}/l^n) = \text{Ext}^\bullet((\mathbb{Z}/l^n)_{\mathcal{X}^{\text{et}}}, (\mathbb{Z}/l^n)_{\mathcal{X}^{\text{et}}})$ and

$$H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l) := (\lim_{\leftarrow} H^\bullet(\mathcal{X}^{\text{et}}, \mathbb{Z}/l^n)) \otimes_{\mathbb{Z}_l} \bar{\mathbb{Q}}_l$$

Again there is an obvious standard spectral sequence converging to $H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l)$.

The morphism of topoi ι induces the corresponding morphism $\iota : \mathcal{X}^{\text{an}} \rightarrow \mathcal{X}^{\text{et}}$ and the functor $\iota^* : \text{Sh}(\mathcal{X}^{\text{et}}) \rightarrow \text{Sh}(\mathcal{X}^{\text{an}})$.

Lemma 10.2. *The morphism of topoi $\iota : \mathcal{X}^{\text{an}} \rightarrow \mathcal{X}^{\text{et}}$ induces an isomorphism $H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l) = H^\bullet(\mathcal{X}^{\text{an}}, \bar{\mathbb{Q}}_l)$. More precisely, there is a natural morphism of standard spectral sequences converging to $H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l)$ and $H^\bullet(\mathcal{X}^{\text{an}}, \bar{\mathbb{Q}}_l)$ respectively, which induces an isomorphism of the corresponding E_1 -complexes.*

Proof. For each $\alpha \in S$ and $n \in \mathbb{Z}$ denote by $(\mathbb{Z}/l^n)_{X_\alpha} \rightarrow G_{\alpha,n}$ the canonical Godement flabby resolution [Go],[FK], pp.129-130. Then naturally $G_n = \{G_{\alpha,n}\}$ is a complex in $\text{Sh}(\mathcal{X}^{\text{et}})$. Moreover, $G_{n+1} \otimes_{\mathbb{Z}/l^{n+1}} \mathbb{Z}/l^n = G_n$. The cohomology $H^\bullet(\mathcal{X}^{\text{et}}, \mathbb{Z}/l^n)$ can be computed using the resolution G_n . In particular the standard spectral sequence converging

to $H^\bullet(\mathcal{X}^{\text{et}}, \mathbb{Z}/l^n)$ is defined by the double complex $\Gamma(G_n)$ which consists of sums of groups $\Gamma(X_\alpha, G_n)$. These double complexes form an inverse system

$$(10.2) \quad \dots \rightarrow \Gamma(G_2) \rightarrow \Gamma(G_1)$$

and the double complex

$$(10.3) \quad \varprojlim \Gamma(G_n) \otimes_{\mathbb{Z}_l} \bar{\mathbb{Q}}_l$$

computes the cohomology $H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l)$. Applying the functor ι^* to the inverse system of complexes $\{G_n\}$ provides the desired morphism of standard spectral sequences for $H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l)$ and $H^\bullet(\mathcal{X}^{\text{an}}, \bar{\mathbb{Q}}_l)$ respectively. This morphism induces an isomorphism of E_1 -terms, because $H^\bullet(X_\alpha^{\text{et}}, \bar{\mathbb{Q}}_l) = H^\bullet(X_\alpha^{\text{an}}, \bar{\mathbb{Q}}_l)$ for each α . \square

So in order to prove the theorem it suffices to show the degeneration of the standard spectral sequence for $H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l)$.

Step 2. For any smooth complex scheme Y we can find a discrete valuation ring $V \subset \mathbb{C}$ whose residue field is the algebraic closure of a finite field, and a smooth morphism $Y_V \rightarrow \text{Spec} V$, such that Y is obtained by extension of scalars from Y_V . Let Y_s be the closed fiber of Y_V . We obtain the diagram of schemes

$$Y \xrightarrow{u} Y_V \xleftarrow{i} Y_s.$$

These morphisms induce isomorphisms

$$H^\bullet(Y^{\text{et}}, \bar{\mathbb{Q}}_l) \xleftarrow{u^*} H^\bullet(Y_V^{\text{et}}, \bar{\mathbb{Q}}_l) \xrightarrow{i^*} H^\bullet(Y_s^{\text{et}}, \bar{\mathbb{Q}}_l).$$

This extends to smooth poset schemes. Namely, we can find V as above and a smooth poset scheme \mathcal{X}_V over $\text{Spec} V$, which gives rise to \mathcal{X} by extension of scalars. Let \mathcal{X}_s again be the closed fiber, which is a smooth poset scheme over $\bar{\mathbb{F}}_q$. Consider the corresponding diagram of poset schemes

$$\mathcal{X} \xrightarrow{u} \mathcal{X}_V \xleftarrow{i} \mathcal{X}_s.$$

Lemma 10.3. *The morphisms u, i induce isomorphisms*

$$H^\bullet(\mathcal{X}^{\text{et}}, \bar{\mathbb{Q}}_l) \xleftarrow{u^*} H^\bullet(\mathcal{X}_V^{\text{et}}, \bar{\mathbb{Q}}_l) \xrightarrow{i^*} H^\bullet(\mathcal{X}_s^{\text{et}}, \bar{\mathbb{Q}}_l).$$

More precisely the morphisms u, i induce morphisms of the standard spectral sequences converging to these groups. And these morphisms induces isomorphisms of the corresponding E_1 -terms.

Proof. The proof is very similar to the proof of Lemma 10.2. Namely one considers the Godement resolution G_n of the constant sheaf \mathbb{Z}/l^n on \mathcal{X}_V and passes to the inverse limit. We omit the details. \square

So it suffices to prove the degeneration of the standard spectral sequence for

$$H^\bullet(\mathcal{X}_s^{\text{et}}, \bar{\mathbb{Q}}_l).$$

Step 3. The geometric Frobenius endomorphism Fr acts on the smooth poset scheme \mathcal{X}_s and hence on the standard spectral sequence which converges to $H^\bullet(\mathcal{X}_s^{\text{et}}, \bar{\mathbb{Q}}_l)$. For each $\alpha \in S$ Fr acts on $H^n(X_{\alpha s}^{\text{et}}, \bar{\mathbb{Q}}_l)$ with eigenvalues θ such that $|\theta| = q^{n/2}$ (Weil conjectures, see [De]).

In the standard spectral sequence each differential d_r for $r \geq 2$ is a map between subquotients of $H^n(X_{\alpha s}^{\text{et}}, \bar{\mathbb{Q}}_l)$ and $H^m(X_{\beta s}^{\text{et}}, \bar{\mathbb{Q}}_l)$ for $n > m$. Hence $d_r = 0$ for $r \geq 2$. This completes the proof of Theorem 10.1. \square

11. DEGENERATION OF HODGE TO DERHAM SPECTRAL SEQUENCE FOR SMOOTH PROJECTIVE POSET SCHEMES.

Definition 11.1. Let \mathcal{X} be a smooth complex projective poset scheme. Recall that the analytic deRham complex $\Omega_{\mathcal{X}^{\text{an}}}^\bullet$ is a resolution of the constant sheaf $\mathbb{C}_{\mathcal{X}^{\text{an}}}$. As in the case of a single smooth variety the "stupid" filtration $F^p \Omega_{\mathcal{X}^{\text{an}}}^\bullet := \bigoplus_{i \geq p} \Omega_{\mathcal{X}^{\text{an}}}^i$ of this deRham complex gives rise to the Hodge-to-deRham spectral sequence converging to $H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C})$.

The following theorem is the poset scheme analogue of the well known degeneration of the Hodge-to-deRham spectral sequence for smooth projective varieties. The proof uses Theorem 10.1 above.

Theorem 11.2. Let \mathcal{X} be a smooth complex projective poset scheme. Then the Hodge-to-deRham spectral sequence degenerates at the E_2 -term. That is $d_2 = d_3 = \dots = 0$. Hence

$$(11.1) \quad H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C}) = \bigoplus_p H^{\bullet-p}(\mathcal{X}^{\text{an}}, \Omega_{\mathcal{X}^{\text{an}}}^p).$$

In particular the map $\mathbb{C}_{\mathcal{X}^{\text{an}}} \rightarrow \mathcal{O}_{\mathcal{X}^{\text{an}}}$ induces a surjection $H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C}) \rightarrow H^\bullet(\mathcal{X}^{\text{an}}, \mathcal{O}_{\mathcal{X}^{\text{an}}}) = H^\bullet(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

The decomposition 11.1 is (contravariant) functorial with respect to morphisms of smooth projective poset schemes.

Proof. The degeneration of the Hodge-to-deRham spectral sequence follows by dimension counting from the isomorphism 11.1. The last assertion of the theorem is obvious. So it suffices to prove 11.1. To compute the cohomology of $\mathbb{C}_{\mathcal{X}^{\text{an}}}$ we may use the Dolbeaut resolution $\mathbb{C}_{\mathcal{X}^{\text{an}}} \xrightarrow{\sim} \Omega_{\mathcal{X}^{\text{an}}}^\bullet \rightarrow \mathcal{A}_{\mathcal{X}^{\text{an}}}^{\bullet\bullet}$, where $\mathcal{A}^{p,q}$ is the sheaf of C^∞ (p, q) -forms. The canonical morphism of complexes $\Omega_{\mathcal{X}^{\text{an}}}^\bullet \leftarrow \Omega_{\mathcal{X}^{\text{an}}}^{\geq p} \rightarrow \Omega_{\mathcal{X}^{\text{an}}}^p$ lifts to a morphism of the corresponding Dolbeaut resolutions $\mathcal{A}^{\bullet\bullet} \leftarrow \mathcal{A}^{\geq p, \bullet} \rightarrow \mathcal{A}^{p, \bullet}$. Thus we obtain the induced morphisms of standard spectral sequences (Definition 9.3) for $\Omega_{\mathcal{X}^{\text{an}}}^\bullet, \Omega_{\mathcal{X}^{\text{an}}}^{\geq p}$ and $\Omega_{\mathcal{X}^{\text{an}}}^p$ respectively.

Using the usual Hodge decomposition for each $X_\alpha \in \mathcal{X}$ we find that the E_1 -term of the standard spectral sequence for $\Omega_{\mathcal{X}^{\text{an}}}^\bullet$ is the direct sum of complexes $E_1^{(p,q)}$, where $E_1^{(p,q)}$ consists of summands $H^{p,q}(X_\alpha, \mathbb{C})$. Certainly the E_1 term of the standard spectral sequence for the complex $\Omega_{\mathcal{X}^{\text{an}}}^{\geq p}$ (resp. $\Omega_{\mathcal{X}^{\text{an}}}^{\leq p}$, resp. $\Omega_{\mathcal{X}^{\text{an}}}^p$) identifies as a direct summand of this complex which consists of summands $H^{\geq p, \bullet}(X_\alpha, \mathbb{C})$ (resp. $H^{\leq p, \bullet}(X_\alpha, \mathbb{C})$, resp. $H^{p, \bullet}(X_\alpha, \mathbb{C})$). By Theorem 10.1 the standard spectral sequence for the complex $\Omega_{\mathcal{X}^{\text{an}}}^\bullet$ degenerates at E_2 . Applying the next lemma we conclude that the standard spectral sequences for these other complexes also degenerate at E_2 . Now using the dimension count we find the isomorphism 11.1, which proves the theorem. \square

Lemma 11.3. *Let $A \rightarrow B$ be a morphism of bounded below double complexes. Denote by $E_r(A)$ and $E_r(B)$ the E_r -terms of the corresponding spectral sequences converging to $H^\bullet(\text{Tot}(A))$ and $H^\bullet(\text{Tot}(B))$ respectively.*

i) Assume that the spectral sequence for B degenerates at $E_r(B)$, i.e. $0 = d_r(B) = d_{r+1}(B) = \dots$ and the induced map of complexes $E_r(A) \rightarrow E_r(B)$ is injective. Then the sequence for A also degenerates at E_r .

ii) Assume that the sequence for A degenerates at E_r and the map $E_r(A) \rightarrow E_r(B)$ is surjective. Then the sequence for B also degenerates at E_r .

Proof. This is obvious. \square

In the proof of the last theorem we also obtained the following result.

Proposition 11.4. *Let \mathcal{X} be a smooth complex projective poset scheme. Then the standard spectral sequences converging to the cohomology of \mathcal{X}^{an} with coefficients respectively in $\Omega_{\mathcal{X}^{\text{an}}}^{\geq p}, \Omega_{\mathcal{X}^{\text{an}}}^{\leq p}, \Omega_{\mathcal{X}^{\text{an}}}^p$ degenerate at E_2 -terms.*

Now using GAGA we derive the corresponding statements in the algebraic category. Namely let \mathcal{X} be a smooth complex projective poset scheme. We consider again the "stupid" filtration $F^p \Omega_{\mathcal{X}}^\bullet := \bigoplus_{i \geq p} \Omega_{\mathcal{X}}^i$ of the algebraic deRham complex. It gives rise to the spectral sequence converging to $H^\bullet(\mathcal{X}^{\text{an}}, \Omega_{\mathcal{X}}^\bullet)$. We also call it "Hodge-to-de Rham".

Theorem 11.5. *Let \mathcal{X} be a smooth complex projective poset scheme.*

a) The (algebraic) Hodge-to-de Rham spectral sequence degenerates at the E_2 -term. That is $d_2 = d_3 = \dots = 0$. Hence

$$(11.2) \quad H^\bullet(\mathcal{X}, \Omega_{\mathcal{X}}^\bullet) = \bigoplus_p H^{\bullet-p}(\mathcal{X}, \Omega_{\mathcal{X}}^p).$$

The decomposition 11.2 is functorial with respect to morphisms of poset schemes.

b) The standard spectral sequences converging to the cohomology of \mathcal{X} with coefficients respectively in $\Omega_{\mathcal{X}}^{\geq p}, \Omega_{\mathcal{X}}^{\leq p}, \Omega_{\mathcal{X}}^p$ degenerate at E_2 -terms.

Proof. a) As in the analytic case everything follows from the isomorphism 11.2 by dimension counting. But this isomorphism 11.2 follows from the isomorphism 11.1 and Subsection 9.1.

b) This follows from Proposition 11.4 and Subsection 9.1. \square

Example 11.6. *Let us give a simple example of a projective poset scheme which is not smooth and for which the standard spectral sequence converging to $H^\bullet(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ does not degenerate at E_2 . Namely, let X be a projective curve which is the union of two projective lines C_1 and C_2 which intersect transversally at 2 points p_1 and p_2 . Then $H^1(X, \mathcal{O}_X)$ has dimension 1. Now take two copies of the curve $X = X_1 = X_2$, and let the poset scheme \mathcal{X} consist of $X_1, X_2, C_1, C_2, p_1, p_2$ with the obvious maps from each of the C 's (resp. p 's) to each of the X 's (resp. C 's). Then a standard spectral sequence converging to $H^\bullet(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ has for the E_1 -term the natural complex*

$$0 \rightarrow H^\bullet(X_1) \oplus H^\bullet(X_2) \rightarrow H^\bullet(C_1) \oplus H^\bullet(C_2) \rightarrow H^\bullet(p_1) \oplus H^\bullet(p_2) \rightarrow 0$$

where $H^\bullet(Y)$ denotes $H^\bullet(Y, \mathcal{O}_Y)$. Let $0 \neq a \in H^1(X, \mathcal{O}_X)$. Then $(a, -a)$ is a nonzero cycle in the above complex and it is not difficult to check that $d_2(a, -a) \neq 0$.

12. CUBICAL HYPERRESOLUTIONS AND DU BOIS SINGULARITIES

Cubical hyperresolutions are poset schemes of a certain type. Here we briefly recall the definition and the main properties of cubical hyperresolutions according to [LNM1335], Ex.1.

For each integer $n \geq -1$ we denote by \square_n^+ the poset which is the product of $n+1$ copies of the poset $\{0, 1\}$. Thus for $n = -1$ the poset \square_{-1}^+ consists of one element and $\square_0^+ = \{0, 1\}$. Let \square_n denote the complement in \square_n^+ of the initial object $(0 \dots 0)$. For $\alpha = (\alpha_0 \dots \alpha_n) \in \square_n^+$ we put $|\alpha| = \alpha_0 + \dots + \alpha_n$.

Definition 12.1. *Let S be a (finite) poset, \mathcal{X} be a reduced separated S -scheme of finite type, and let \mathcal{Z} be a reduced $\square_1^+ \times S$ -scheme. We call \mathcal{Z} a 2-resolution of \mathcal{X} if for each $\beta \in S$ the commutative diagram*

$$\begin{array}{ccc} Z_{11\beta} & \rightarrow & Z_{01\beta} \\ \downarrow & & \downarrow f \\ Z_{10\beta} & \rightarrow & Z_{00\beta} \end{array}$$

has the following properties:

- 1) it is a cartesian square,
- 2) $Z_{00\beta} = X_\beta$,
- 3) $Z_{01\beta}$ is smooth,
- 4) horizontal arrows are closed embeddings,
- 5) the morphism f is proper,

6) $Z_{10\beta}$ contains the discriminant of f . In other words f induces an isomorphism $f : Z_{01\beta} \setminus Z_{11\beta} \xrightarrow{\sim} Z_{00\beta} \setminus Z_{10\beta}$.

Definition 12.2. Fix a poset S and an integer $r \geq 1$. Assume that for each $1 \leq n \leq r$ we are given an $\square_n^+ \times S$ -scheme \mathcal{X}^n so that the $\square_{n-1}^+ \times S$ schemes $\mathcal{X}_{00\bullet}^{n+1}$ and $\mathcal{X}_{1\bullet}^n$ are equal. We define by induction on r an $\square_r^+ \times S$ -scheme $\mathcal{Z} = \text{rd}(\mathcal{X}^1, \mathcal{X}^2, \dots, \mathcal{X}^r)$, which we call the reduction of $(\mathcal{X}^1, \dots, \mathcal{X}^r)$. Namely, if $r = 1$ we put $\mathcal{Z} = \mathcal{X}^1$. If $r = 2$ we define

$$Z_{\alpha\beta} = \begin{cases} X_{0\beta}^1, & \text{if } \alpha = (00), \\ X_{\alpha\beta}^2, & \text{if } \alpha \in \square_1 \end{cases}$$

for all $\beta \in \square_0^+ \times S$. For $r > 2$ we put

$$\mathcal{Z} = \text{rd}(\text{rd}(\mathcal{X}^1, \dots, \mathcal{X}^{r-1}), \mathcal{X}^r).$$

Definition 12.3. Let S be a poset and \mathcal{X} be an S -scheme. An augmented cubical hyperresolution of \mathcal{X} is an $\square_r^+ \times S$ -scheme \mathcal{Z}^+ such that

$$\mathcal{Z}^+ = \text{rd}(\mathcal{X}^1, \dots, \mathcal{X}^r),$$

where

- 1) \mathcal{X}^1 is a 2-resolution of \mathcal{X} ,
- 2) for each $1 \leq n \leq r$, \mathcal{X}^{n+1} is a 2-resolution of $\mathcal{X}_{1\bullet}^n$, and
- 2) Z_α is smooth for each $\alpha \in \square_r$.

We will call the \square_r -scheme $\mathcal{Z} = \mathcal{Z}^+ \setminus Z_{(0, \dots, 0)}$ a cubical hyperresolution of \mathcal{X} . It comes with the augmentation morphism of poset schemes $\pi : \mathcal{Z} \rightarrow \mathcal{X}$, which is compatible with the projection of posets $\square_r \times S \rightarrow S$.

Theorem 12.4. Assume that the base field k has characteristic zero. Let S be a poset and \mathcal{X} be a separated reduced S -scheme of finite type. Then there exists an augmented cubical hyperresolution \mathcal{Z} of \mathcal{X} , such that $\dim Z_\alpha \leq \dim \mathcal{X} - |\alpha| + 1$.

Proposition 12.5. Let S be a poset, \mathcal{X} an S -scheme and \mathcal{Z} an $\square_r^+ \times S$ -scheme, which is an augmented cubical hyperresolution of \mathcal{X} . Then for each $\alpha \in S$ the \square_r^+ -scheme $\mathcal{Z}_{\bullet\alpha}$ is an augmented cubical hyperresolution of X_α .

We refer the reader to [LNM1335], Ex.1, Thm.2.15, Prop.2.14 for the proof of the above theorem and proposition and also for the study of the category of cubical hyperresolutions of S -schemes.

Remark 12.6. Let X be a reduced separated complex scheme of finite type and let $\pi : \mathcal{Z} \rightarrow X$ be a cubical hyperresolution. Then $\mathbf{R}\pi_*^{\text{an}} \mathbb{C}_{\mathcal{Z}^{\text{an}}} = \mathbb{C}_{X^{\text{an}}}$. This follows from [LNM1335], Ex.1, Thm.6.1.

Definition 12.7. Let X be a reduced separated scheme of finite type over a field of characteristic zero. Choose its cubical hyperresolution $\pi : \mathcal{Z} \rightarrow X$. We say that X has Du Bois singularities (X is Du Bois, for short) if the adjunction morphism $\mathcal{O}_X \rightarrow \mathbf{R}\pi_*\mathcal{O}_{\mathcal{Z}}$ is a quasi-isomorphism.

Remark 12.8. The complex $\mathbf{R}\pi_*\mathcal{O}_{\mathcal{Z}} \in D(X)$ is independent (up to a quasi-isomorphism) on the choice of a hyperresolution of X ([LNM1335], Ex.3). So the notion of Du Bois singularities is well defined.

Remark 12.9. If X has rational singularities (for example X is smooth), then X is Du Bois. It was conjectured by Kollar [Ko] and recently proved by Kollar and Kovac [KoKov] that if X has log canonical singularities, then X is Du Bois.

Theorem 12.10. Let X be a reduced separated scheme of finite type over a field of characteristic zero. Choose its hyperresolution $\pi : \mathcal{Z} \rightarrow X$. Assume that the adjunction map $\mathcal{O}_X \rightarrow \mathbf{R}\pi_*\mathcal{O}_{\mathcal{Z}}$ has a left inverse. Then X is Du Bois (i.e. this map is a quasi-isomorphism).

Proof. See [Kov]. □

The notion of Du Bois singularities characterizes the existence of categorical resolutions by smooth poset schemes as is shown in the next theorem.

Theorem 12.11. Let X be a reduced scheme of finite type over a field of characteristic zero. Then there exists a categorical resolution of X by a smooth poset scheme (Definition 7.1) if and only if X has Du Bois singularities.

Proof. One direction is clear: if X has Du Bois singularities and $\pi : \mathcal{Z} \rightarrow X$ is its hyperresolution then by Proposition 7.2 π is a categorical resolution of X by the smooth poset scheme \mathcal{Z} .

Vice versa, assume that S is a poset, \mathcal{X} is a smooth S -scheme and $\sigma : \mathcal{X} \rightarrow X$ is a categorical resolution. Consider the augmented $S^+ := S \cup \{0\}$ -scheme \mathcal{X}^+ defined by σ (so that $X_0 = X$). A choice of a hyperresolution of $\pi : \mathcal{Y} \rightarrow \mathcal{X}^+$ induces a commutative diagram of poset schemes

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow \tilde{\sigma} & & \downarrow \sigma \\ \mathcal{Y}_0 & \xrightarrow{\pi_0} & X \end{array}$$

which is compatible with the diagram of projections of posets

$$\begin{array}{ccc} \square_n \times S & \rightarrow & S \\ \downarrow & & \downarrow \\ \square_n & \rightarrow & \{0\} \end{array}$$

and such that π_0 (and π) are hyperresolutions (Proposition 12.5).

By our assumption the adjunction map $\mathcal{O}_X \rightarrow \mathbf{R}\sigma_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and we want to prove that the adjunction morphism $\mathcal{O}_X \rightarrow \mathbf{R}(\pi_0)_*\mathcal{O}_{\mathcal{Y}_0}$ is an isomorphism. By Theorem 12.10 it suffices to prove that this last map has a left inverse.

Since the poset scheme \mathcal{X} is smooth we conclude by Remark 12.9, Proposition 12.5 and Lemma 6.3 that the map $\mathcal{O}_{\mathcal{X}} \rightarrow \mathbf{R}\pi_*\mathcal{O}_{\mathcal{Y}}$ is an isomorphism. Thus the adjunction map $\mathcal{O}_X \rightarrow \mathbf{R}(\sigma \cdot \pi)_*\mathcal{O}_{\mathcal{Y}} = \mathbf{R}(\pi_0 \cdot \tilde{\sigma})\mathcal{O}_{\mathcal{Y}}$ is an isomorphism. But this last map is the composition of the adjunction maps $\mathcal{O}_X \rightarrow \mathbf{R}(\pi_0)_*\mathcal{O}_{\mathcal{Y}_0} \rightarrow \mathbf{R}(\pi_0)_* \cdot \mathbf{R}(\tilde{\sigma})_*\mathcal{O}_{\mathcal{Y}}$. Hence the map $\mathcal{O}_X \rightarrow \mathbf{R}(\pi_0)_*\mathcal{O}_{\mathcal{Y}_0}$ has a left inverse. This proves the theorem. \square

Cubical hyperresolutions give more: one can define the de Rham complex of a singular algebraic variety X . Namely, choose a hyperresolution $\pi : \mathcal{Z} \rightarrow X$ and define the de Rham-Du Bois complex $\underline{\Omega}_X^\bullet := \mathbf{R}\pi_*\Omega_{\mathcal{Z}}^\bullet$. This complex consists of \mathcal{O}_X -modules and has the differential which is a differential operator of order 1. It has coherent cohomology and is well defined (independent of the choice of a hyperresolution) up to a quasi-isomorphism in the appropriate derived category [LNM1335], Ex.3. There exists a canonical morphism of filtered complexes from the usual de Rham complex Ω_X^\bullet to $\underline{\Omega}_X^\bullet$ which is a quasi-isomorphism if X is smooth.

If X is a reduced separated complex scheme, then the analytization $(\underline{\Omega}_X^\bullet)^{\text{an}} = \underline{\Omega}_{X^{\text{an}}}^\bullet$ is a resolution of the constant sheaf $\mathbb{C}_{X^{\text{an}}}$.

The stupid filtration of the complex $\Omega_{\mathcal{Z}}^\bullet$ induces a filtration on the de Rham-Du Bois complex and $\underline{\Omega}_X^\bullet$ is well defined even as a filtered complex. The associated graded pieces are $\underline{\Omega}_X^i := \text{gr}^i \underline{\Omega}_X^\bullet = \mathbf{R}\pi_*\Omega_{\mathcal{Z}}^i$. If X is proper then this filtration induces the Hodge filtration on $H^\bullet(X^{\text{an}}, \mathbb{C})$.

We will prove in Theorem 14.1 below that for a reduced complex projective scheme X the filtered complex $\underline{\Omega}_X^\bullet$ can be defined as $\mathbf{R}\sigma_*\Omega_{\mathcal{X}}^\bullet$, where \mathcal{X} is a smooth complex projective poset scheme and $\sigma : \mathcal{X} \rightarrow X$ is a morphism such that $\mathbf{R}\sigma_*^{\text{an}}\mathbb{C}_{\mathcal{X}^{\text{an}}} = \mathbb{C}_{X^{\text{an}}}$.

13. EXAMPLES OF CATEGORICAL RESOLUTIONS BY SMOOTH POSET SCHEMES

Let Y be a reducible scheme with irreducible components Y_1, \dots, Y_n . Assume that for each $1 \leq k \leq n$ and each subset $\alpha = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ the scheme

$$X_\alpha := \bigcap_{j=1}^k Y_{i_j}$$

is smooth. (In particular the components Y_i are smooth.) Let S be the poset of nonempty subsets of $\{1, \dots, n\}$ with the natural partial ordering by inclusion. Let $\mathcal{X} = \{X_\alpha\}$ be

the corresponding smooth poset scheme with the maps $f_{\alpha\beta} : X_\alpha \rightarrow X_\beta$ being the obvious inclusions. Let $\pi : \mathcal{X} \rightarrow Y$ be the natural morphism.

Proposition 13.1. *The functor $\mathbf{L}\pi^* : D(Y) \rightarrow D(\mathcal{X})$ is a categorical resolution of singularities, i.e. the functor*

$$\mathbf{L}\pi^* : \text{Perf}(Y) \rightarrow \text{Perf}(\mathcal{X})$$

is full and faithful.

Proof. By Proposition 7.2 we may assume that Y is affine and we only need to prove that the map $\text{Ext}(\mathcal{O}_Y, \mathcal{O}_Y) \rightarrow \text{Ext}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ is an isomorphism.

We have $\text{Ext}^i(\mathcal{O}_Y, \mathcal{O}_Y) = 0$ for $i \neq 0$. On the other hand we have the obvious complex in $C(\mathcal{X})$

$$C(\mathcal{O}_{\mathcal{X}}) := \dots \rightarrow \bigoplus_{|\alpha|=2} j_{\alpha+}(\mathcal{O}_{\mathcal{X}})_\alpha \rightarrow \bigoplus_{|\beta|=1} j_{\beta+}(\mathcal{O}_{\mathcal{X}})_\beta \rightarrow 0,$$

which is a resolution of $\mathcal{O}_{\mathcal{X}}$. Since all schemes X_α are affine we have $\text{Hom}(C(\mathcal{O}_{\mathcal{X}}), \mathcal{O}_{\mathcal{X}}) = \mathbf{R}\text{Hom}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ (Example 8.5). But $\text{Hom}(C(\mathcal{O}_{\mathcal{X}}), \mathcal{O}_{\mathcal{X}})$ is the complex

$$0 \rightarrow \bigoplus_{|\beta|=1} H^0(X_\beta, \mathcal{O}_{X_\beta}) \rightarrow \bigoplus_{|\alpha|=2} H^0(X_\alpha, \mathcal{O}_{X_\alpha}) \rightarrow \dots$$

which is quasi-isomorphic to $H^0(Y, \mathcal{O}_Y)$. □

13.1. Categorical resolution of the cone over a plane cubic. Here we show how smooth poset schemes can be used to construct a categorical resolution of the simplest nonrational singularity - the cone over a smooth plain cubic.

Let $C \subset \mathbb{P}^2$ be a smooth curve of degree 3 (and genus 1) and $Y \subset \mathbb{P}^3$ be the projective cone over C . So Y is a cubic surface with a singular point p - the vertex of the cone. We have

$$H^i(Y, \mathcal{O}_Y) = \begin{cases} k, & \text{if } i=0 \\ 0, & \text{otherwise.} \end{cases}$$

Let $f : X \rightarrow Y$ be the blowup of the vertex, so that X is a smooth ruled surface over the curve C . Denote by $i : E = f^{-1}(p) \hookrightarrow X$ the inclusion of the exceptional divisor. We have

$$H^i(X, \mathcal{O}_X) = \begin{cases} k, & \text{if } i=0,1 \\ 0, & \text{otherwise,} \end{cases}$$

and the pullback map $i^* : H^\bullet(X, \mathcal{O}_X) \rightarrow H^\bullet(E, \mathcal{O}_E)$ is an isomorphism.

Consider the following smooth poset scheme \mathcal{X}

$$\begin{array}{ccc} E & \rightarrow & X \\ \downarrow & & \\ q & & \end{array}$$

where $q = \text{Speck}$, and the map $E \rightarrow X$ is the embedding i . Denote by $\pi : \mathcal{X} \rightarrow Y$ the obvious morphism which extends the blowup $f : X \rightarrow Y$.

Proposition 13.2. $\mathbf{L}\pi^* : D(Y) \rightarrow D(\mathcal{X})$ is a categorical resolution of singularities, i.e. the functor

$$\mathbf{L}\pi^* : \text{Perf}(Y) \rightarrow \text{Perf}(\mathcal{X})$$

is full and faithful.

Proof. Note that the map π is an isomorphism away from the point $p \in Y$. So we may replace Y by the corresponding affine cone Y_0 over C , $f_0 : X_0 \rightarrow Y_0$ is still the blowup of the vertex and the rest is the same. Denote the corresponding poset scheme by \mathcal{X}_0 . Then it suffices to prove that the map $H^\bullet(Y_0, \mathcal{O}_{Y_0}) \rightarrow H^\bullet(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is an isomorphism. We have $H^i(Y_0, \mathcal{O}_{Y_0}) = 0$ for $i \neq 0$. To compute $H(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ we may use the spectral sequence as in Example 8.5. Then the E_1 -term is the sum of the two complexes:

$$k \oplus \Gamma(X_0, \mathcal{O}_{X_0}) \rightarrow \Gamma(E, \mathcal{O}_E), \quad \text{and} \quad H^1(X_0, \mathcal{O}_{X_0}) \rightarrow H^1(E, \mathcal{O}_E).$$

The second map is an isomorphism, and the first one is surjective with the kernel $\Gamma(Y_0, \mathcal{O}_{Y_0})$. \square

In view of Theorem 12.11 above the last example is a special case of the following result of Du Bois [DuB], Prop. 4.13.

Proposition 13.3. Let $W \subset \mathbf{P}^m$ be a smooth variety such that for all $i > 0$ and $n > 0$ the following holds

$$H^i(W, \mathcal{O}(n)) = 0.$$

Then the cone over W has Du Bois singularities.

Remark 13.4. In fact, using the same construction as in the above example of the cone over a smooth cubic curve it is easy to see that the condition in the last proposition is necessary for the cone over W to be Du Bois. For example if $W \subset \mathbb{P}^2$ is a smooth curve of degree ≥ 4 , then the cone over W is not Du Bois.

Some other examples of Du Bois singularities are listed in [St]. For example if X is a reduced curve, then X is Du Bois if and only if at every singular point of X the branches are smooth and their tangent directions are independent.

14. DESCENT FOR DU BOIS SINGULARITIES

Theorem 14.1. *Let X be a reduced complex projective scheme. Let \mathcal{X} be a smooth complex projective poset scheme and $\sigma : \mathcal{X} \rightarrow X$ be a morphism such that the adjunction map $\mathbb{C}_{X^{\text{an}}} \rightarrow \mathbf{R}\sigma_*^{\text{an}} \mathbb{C}_{\mathcal{X}^{\text{an}}}$ is a quasi-isomorphism. Consider the direct image $\mathbf{R}\sigma_* \Omega_{\mathcal{X}}^{\bullet}$. This complex has a filtration induced by the stupid filtration of the de Rham complex $\Omega_{\mathcal{X}}^{\bullet}$. Then there exists a natural morphism of filtered complexes*

$$\tau : \underline{\Omega}_X^{\bullet} \rightarrow \mathbf{R}\sigma_* \Omega_{\mathcal{X}}^{\bullet}$$

which is a quasi-isomorphism. In particular, the map

$$\text{gr}^i \tau : \underline{\Omega}_X^i \xrightarrow{\sim} \mathbf{R}\sigma_* \Omega_{\mathcal{X}}^i$$

is a quasi-isomorphism for all $i \geq 0$. So if X has Du Bois singularities, then $\underline{\Omega}_X^0 \simeq \mathcal{O}_X \simeq \mathbf{R}\sigma_* \mathcal{O}_{\mathcal{X}}$, i.e. the functor $\mathbf{L}\sigma^* : D(X) \rightarrow D(\mathcal{X})$ is a categorical resolution of singularities.

Proof. As in the proof of Theorem 12.11 choose a commutative diagram

$$(14.1) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow \tilde{\sigma} & & \downarrow \sigma \\ \mathcal{Y}_0 & \xrightarrow{\pi_0} & X \end{array}$$

where π_0 is a hyperresolution and for each scheme $X_{\alpha} \in \mathcal{X}$ the induced morphism $\pi : \pi^{-1}(X_{\alpha}) \rightarrow X_{\alpha}$ is also a hyperresolution.

Since each X_{α} is smooth we have the quasi-isomorphism of filtered complexes $\Omega_{\mathcal{X}}^{\bullet} \xrightarrow{\sim} \mathbf{R}\pi_* \Omega_{\mathcal{Y}}^{\bullet}$. It follows that $\mathbf{R}\sigma_* \Omega_{\mathcal{X}}^{\bullet} \simeq \mathbf{R}(\sigma \cdot \pi)_* \Omega_{\mathcal{Y}}^{\bullet} = \mathbf{R}(\pi_0 \cdot \tilde{\sigma})_* \Omega_{\mathcal{Y}}^{\bullet}$. On the other hand by definition $\mathbf{R}(\pi_0)_* \Omega_{\mathcal{Y}_0}^{\bullet} = \underline{\Omega}_X^{\bullet}$. Hence the adjunction morphism $\theta : \Omega_{\mathcal{Y}_0}^{\bullet} \rightarrow \mathbf{R}\tilde{\sigma}_* \Omega_{\mathcal{Y}}^{\bullet}$ induces the desired morphism of filtered complexes

$$(14.2) \quad \tau : \underline{\Omega}_X^{\bullet} = \mathbf{R}(\pi_0)_* \Omega_{\mathcal{Y}_0}^{\bullet} \xrightarrow{\mathbf{R}(\pi_0)_* \theta} \mathbf{R}(\pi_0 \cdot \tilde{\sigma})_* \Omega_{\mathcal{Y}}^{\bullet} \simeq \mathbf{R}\sigma_* \Omega_{\mathcal{X}}^{\bullet}.$$

We will prove that for each i the map

$$\text{gr}^i \tau : \underline{\Omega}_X^i \rightarrow \mathbf{R}\sigma_* \Omega_{\mathcal{X}}^i$$

is a quasi-isomorphism (hence τ is a quasi-isomorphism).

Lemma 14.2. *For each i the morphism $\text{gr}^i \tau$ induces an isomorphism on the hypercohomology*

$$H^{\bullet}(\text{gr}^i \tau) : H^{\bullet}(X, \underline{\Omega}_X^i) \rightarrow H^{\bullet}(X, \mathbf{R}\sigma_* \Omega_{\mathcal{X}}^i).$$

Proof. Note that the map $H^{\bullet}(\text{gr}^i \tau)$ coincides with the inverse image map $H^{\bullet}(\mathcal{Y}_0, \Omega_{\mathcal{Y}_0}^i) \rightarrow H^{\bullet}(\mathcal{Y}, \Omega_{\mathcal{Y}}^i) = H^{\bullet}(X, \mathbf{R}\sigma_* \Omega_{\mathcal{X}}^i)$.

The diagram 14.1 induces the corresponding diagram of analytic spaces

$$(14.3) \quad \begin{array}{ccc} \mathcal{Y}^{\text{an}} & \xrightarrow{\pi^{\text{an}}} & \mathcal{X}^{\text{an}} \\ \downarrow \tilde{\sigma}^{\text{an}} & & \downarrow \sigma^{\text{an}} \\ \mathcal{Y}_0^{\text{an}} & \xrightarrow{\pi_0^{\text{an}}} & X^{\text{an}} \end{array}$$

By Subsection 9.1 it suffices to show that the corresponding inverse image map $H^\bullet(\text{gr}^i \tau^{\text{an}}) : H^\bullet(\mathcal{Y}_0^{\text{an}}, \Omega_{\mathcal{Y}_0^{\text{an}}}^i) \rightarrow H^\bullet(\mathcal{Y}^{\text{an}}, \Omega_{\mathcal{Y}^{\text{an}}}^i)$ is an isomorphism.

Since π_0 and π are cubical hyperresolutions we have $\mathbf{R}(\pi_0^{\text{an}})_* \mathbb{C}_{\mathcal{Y}_0^{\text{an}}} = \mathbb{C}_{X^{\text{an}}}$ and $\mathbf{R}(\pi^{\text{an}})_* \mathbb{C}_{\mathcal{Y}^{\text{an}}} = \mathbb{C}_{\mathcal{X}^{\text{an}}}$. Thus by our assumption $\mathbf{R}(\sigma^{\text{an}} \cdot \pi^{\text{an}})_* \mathbb{C}_{\mathcal{Y}^{\text{an}}} = \mathbb{C}_{X^{\text{an}}}$. As in the case of the sheaves Ω^i we obtain a natural morphism

$$\tau^c : \mathbf{R}(\pi_0^{\text{an}})_* \mathbb{C}_{\mathcal{Y}_0^{\text{an}}} \rightarrow \mathbf{R}\sigma_*^{\text{an}} \mathbb{C}_{\mathcal{X}^{\text{an}}}$$

which is a quasi-isomorphism (both sides are quasi-isomorphic to $\mathbb{C}_{X^{\text{an}}}$). Hence the map

$$H^\bullet(\mathcal{Y}_0^{\text{an}}, \mathbb{C}) \xrightarrow{H^\bullet(\tau^c)} H^\bullet(\mathcal{X}^{\text{an}}, \mathbb{C}) = H^\bullet(\mathcal{Y}^{\text{an}}, \mathbb{C})$$

is an isomorphism.

By Theorem 11.2

$$(14.4) \quad H^\bullet(\mathcal{Y}_0^{\text{an}}, \mathbb{C}) = \bigoplus_i H^{\bullet-i}(\mathcal{Y}_0^{\text{an}}, \Omega_{\mathcal{Y}_0^{\text{an}}}^i).$$

and similarly for \mathcal{Y} . The map $H^\bullet(\tau^c)$ respects this decomposition and its restriction to the i -th summand is the map $H^\bullet(\text{gr}^i \tau^{\text{an}})$. It follows that $H^\bullet(\text{gr}^i \tau^{\text{an}})$ is also an isomorphism. This proves the lemma. \square

Lemma 14.3. *Let Y be a complex projective scheme with an ample line bundle L . Let $u : K_1 \rightarrow K_2$ be a morphism of complexes in $D^b(\text{coh} Y)$. Assume that for all $n \gg 0$ the map u induces an isomorphism of the hypercohomology*

$$H^\bullet(Y, K_1 \otimes L^n) \xrightarrow{\sim} H^\bullet(Y, K_2 \otimes L^n).$$

Then u is a quasi-isomorphism.

Proof. See Lemma 3.4 in [LNM1335] (p.139). \square

We will prove that the morphism $\text{gr} \tau^i$ satisfies the assumptions of Lemma 14.3, which will prove the theorem.

Proposition 14.4. *Let L be an ample line bundle on X . Then for any $n \geq 1$ the map $\text{gr}^i \tau \otimes L^n : \text{gr}^i \tau : \underline{\Omega}_X^i \otimes L^n \rightarrow (\mathbf{R}\sigma_* \Omega_{\mathcal{X}}^i) \otimes L^n$ induces an isomorphism on hypercohomology*

$$H^\bullet(X, \underline{\Omega}_X^i \otimes L^n) \rightarrow H^\bullet(X, (\mathbf{R}\sigma_* \Omega_{\mathcal{X}}^i) \otimes L^n).$$

Proof. We prove the proposition by induction on the dimension of X . If $\dim X = 0$, then the statement is equivalent to Lemma 14.2.

We denote by L also the pullbacks of L to the smooth poset schemes \mathcal{X} and \mathcal{Y}_0 . By the projection formula it suffices to prove that the natural map

$$H^\bullet(X, \mathbf{R}\pi_{0*}(\Omega_{\mathcal{Y}_0}^i \otimes L^n)) \rightarrow H^\bullet(X, \mathbf{R}\sigma_*(\Omega_{\mathcal{X}}^i \otimes L^n))$$

is an isomorphism.

Lemma 14.5. *Let Y be a smooth variety, $B \subset Y$ - a smooth divisor, and M - the corresponding line bundle. Then for each $i \geq 1$ we have the exact sequences*

$$\begin{aligned} 0 \rightarrow \Omega_Y^i \rightarrow M \otimes \Omega_Y^i \rightarrow M \otimes \Omega_Y^i \otimes \mathcal{O}_B \rightarrow 0, \\ 0 \rightarrow \Omega_B^{i-1} \rightarrow M \otimes \Omega_Y^i \otimes \mathcal{O}_B \rightarrow M \otimes \Omega_B^i \rightarrow 0. \end{aligned}$$

These sequences are functorial with respect to the pair (Y, B) .

Proof. [LNM1335], p.136. □

Let $D \subset X$ be a general divisor corresponding to L^n for $n \geq 1$. Let

$$(14.5) \quad \begin{array}{ccc} \mathcal{Z} & \xrightarrow{\pi} & \mathcal{W} \\ \downarrow \tilde{\sigma} & & \downarrow \sigma \\ \mathcal{Z}_0 & \xrightarrow{\pi_0} & D \end{array}$$

be the restriction of the diagram 14.1 to D . Since D is general this diagram has similar properties: \mathcal{W} is a smooth projective poset scheme, π_0 is a hyperresolution, and for each scheme $W_\alpha \in \mathcal{W}$ the induced morphism $\pi : \pi^{-1}(W_\alpha) \rightarrow W_\alpha$ is also a hyperresolution. Also the adjunction morphism $\mathbb{C}_{D^{\text{an}}} \rightarrow \mathbb{R}\sigma_*^{\text{an}} \mathbb{C}_{\mathcal{W}^{\text{an}}}$ is a quasi-isomorphism.

The exact sequences in the last lemma give rise to similar exact sequences on poset schemes \mathcal{X} and \mathcal{Y}_0 respectively. Namely, we have

$$(14.6) \quad \begin{aligned} 0 \rightarrow \Omega_{\mathcal{Y}_0}^i \rightarrow L^n \otimes \Omega_{\mathcal{Y}_0}^i \rightarrow L^n \otimes \Omega_{\mathcal{Y}_0}^i \otimes \mathcal{O}_{\mathcal{Z}_0} \rightarrow 0, \\ 0 \rightarrow \Omega_{\mathcal{Z}_0}^{i-1} \rightarrow L^n \otimes \Omega_{\mathcal{Y}_0}^i \otimes \mathcal{O}_{\mathcal{Z}_0} \rightarrow L^n \otimes \Omega_{\mathcal{Z}_0}^i \rightarrow 0, \end{aligned}$$

and

$$(14.7) \quad \begin{aligned} 0 \rightarrow \Omega_{\mathcal{X}}^i \rightarrow L^n \otimes \Omega_{\mathcal{X}}^i \rightarrow L^n \otimes \Omega_{\mathcal{X}}^i \otimes \mathcal{O}_{\mathcal{W}} \rightarrow 0, \\ 0 \rightarrow \Omega_{\mathcal{W}}^{i-1} \rightarrow L^n \otimes \Omega_{\mathcal{X}}^i \otimes \mathcal{O}_{\mathcal{W}} \rightarrow L^n \otimes \Omega_{\mathcal{W}}^i \rightarrow 0. \end{aligned}$$

We now push forward these diagrams 14.6 and 14.7 by the functors $\mathbf{R}\pi_{0*}$ and $\mathbf{R}\sigma_*$ respectively. By functoriality we have a morphism between the resulting exact triangles on X . On the hypercohomology this morphism induces an isomorphism in the term $\Omega_{\mathcal{Y}_0}^i$ by Lemma 14.2. By induction it also induces similar isomorphisms in the terms $\Omega_{\mathcal{Z}_0}^{i-1}$ and $L^n \otimes \Omega_{\mathcal{Z}_0}^i$. Hence it induces an isomorphism of hypercohomology in the term $L^n \otimes \Omega_{\mathcal{Y}_0}^i \otimes \mathcal{O}_{\mathcal{Z}_0}$ and thus also in the term $L^n \otimes \Omega_{\mathcal{Y}_0}^i$ which proves the proposition and the theorem. □

□

Part 3. Appendix

15. COHERATOR AND THE FUNCTORS $\mathbf{L}f^*, \mathbf{R}f_*$

Probably this appendix contains nothing new but we decided to put together some "well known" facts for convenience.

Let X be a quasi-compact separated scheme. As usual $QcohX$ denotes the category of quasi-coherent sheaves on X , $C(X) = C(QcohX)$ - the category of complexes over $QcohX$, $D(X) = D(QcohX)$ - the derived category. We also consider the category Mod_X of all \mathcal{O}_X -modules, its category of complexes $C(\text{Mod}_X)$ and the corresponding derived category $D(\text{Mod}_X)$. Let $C_{qc}(\text{Mod}_X) \subset C(\text{Mod}_X)$, $D_{qc}(\text{Mod}_X) \subset D(\text{Mod}_X)$ be the full subcategories of complexes with quasi-coherent cohomologies.

Both $QcohX$ and Mod_X are Grothendieck categories.

The obvious exact functor $\phi : QcohX \rightarrow \text{Mod}_X$ preserves finite limits and arbitrary colimits. It has a left-exact right adjoint functor $Q_X = Q : \text{Mod}_X \rightarrow QcohX$ - the *coherator*. The functor Q preserves arbitrary limits and injective objects. The induced functor $Q : C(\text{Mod}_X) \rightarrow C(X)$ preserves h-injectives. One defines the right derived functor $\mathbf{R}Q : D(\text{Mod}_X) \rightarrow D(X)$ using the h-injectives.

Proposition 15.1. *The functors ϕ , $\mathbf{R}Q$ induce mutually inverse equivalences of categories*

$$\phi : D(X) \rightarrow D_{qc}(\text{Mod}_X), \quad \mathbf{R}Q : D_{qc}(\text{Mod}_X) \rightarrow D(X).$$

Proof. See for example [AlJeLi], Prop.1.3. □

Lemma 15.2. *The functor $\phi : C(X) \rightarrow C(\text{Mod}_X)$ preserves h-flats.*

Proof. Let $F \in C(X)$ be h-flat, $N \in C(\text{Mod}_X)$ be acyclic, $x \in X$. We need to show that the complex of \mathcal{O}_x -modules $(F \otimes_{\mathcal{O}_X} N)_x = F_x \otimes_{\mathcal{O}_x} N_x$ is acyclic. Let $i : \text{Spec} \mathcal{O}_x \rightarrow X$ be the inclusion and $\tilde{N}_x \in C(Qcoh(\text{Spec} \mathcal{O}_x))$ be the sheafification of the acyclic complex N_x of \mathcal{O}_x -modules. Then $i_* \tilde{N}_x$ is an acyclic complex of quasi-coherent sheaves on X . Hence the complex $F \otimes_{\mathcal{O}_X} i_* \tilde{N}_x$ is also acyclic. Thus $F_x \otimes_{\mathcal{O}_x} N_x = (F \otimes_{\mathcal{O}_X} i_* \tilde{N}_x)_x$ is also acyclic. □

Let $f : X \rightarrow Y$ be a quasi-compact separated morphism of quasi-compact separated schemes. One defines the derived functors

$$\mathbf{L}f^* : D(\text{Mod}_Y) \rightarrow D(\text{Mod}_X), \quad \mathbf{R}f_* : D(\text{Mod}_X) \rightarrow D(\text{Mod}_Y),$$

using h-flats and h-injectives in $C(\text{Mod}_Y)$ and $C(\text{Mod}_X)$ respectively [Sp].

We can also define the derived functor $\mathbf{L}f^* : D(Y) \rightarrow D(X)$ using the h-flats in $C(Y)$ (There are enough h-flats in $C(Y)$ [AlJeLi], Prop.1.1).

Lemma 15.3. *There exists a natural isomorphism of functors*

$$\mathbf{L}f^* \cdot \phi_Y = \phi_X \cdot \mathbf{L}f^* : D(Y) \rightarrow D(\text{Mod}_X).$$

Proof. Let $F \in D(Y)$ be h-flat. Then $\phi_X \cdot \mathbf{L}f^*(F) = \phi_X \cdot f^*(F)$. On the other hand $\phi_Y(F)$ is h-flat by Lemma 15.2. Hence $\mathbf{L}f^* \cdot \phi_Y(F) = f^* \cdot \phi_Y(F) = \phi_X \cdot f^*(F)$. \square

Proposition 15.4. *a). The functors $(\mathbf{L}f^*, \mathbf{R}f_*)$ between $D(\text{Mod}_Y)$ and $D(\text{Mod}_X)$ are adjoint.*

b). These functors preserve the subcategories $D_{\text{qc}}(\text{Mod}_Y)$ and $D_{\text{qc}}(\text{Mod}_X)$.

Proof. a). It is [Sp], Prop.6.7. b). For the functor $\mathbf{L}f^*$ it follows from Proposition 15.1 and Lemma 15.3 and for the functor $\mathbf{R}f_*$ it is proved for example in [BoVdB], Thm.3.3.3 for the functor $\mathbf{R}f_*$. \square

The functors $f^* : Qcoh Y \rightarrow Qcoh X$, $f_* : Qcoh X \rightarrow Qcoh Y$ are well defined and clearly $f^* \cdot \phi_Y = \phi_X \cdot f^*$. Hence also $f_* \cdot Q_X = Q_Y \cdot f_*$ by adjunction. One defines the derived functor

$$\mathbf{R}f_* : D(X) \rightarrow D(Y)$$

using h-injectives in $C(X)$.

Proposition 15.5. *There exist a natural isomorphism of functor*

$$\mathbf{R}f_* \cdot \mathbf{R}Q_X \simeq \mathbf{R}Q_Y \cdot \mathbf{R}f_* : D_{\text{qc}}(\text{Mod}_X) \rightarrow D(Y).$$

Proof. Let $I \in D_{\text{qc}}(\text{Mod}_X)$ be h-injective. Then $\mathbf{R}Q_X(I) = Q_X(I)$ is h-injective in $D(X)$. Hence $\mathbf{R}f_* \cdot \mathbf{R}Q_X(I) = f_* \cdot Q_X(I)$. also $\mathbf{R}f_*(I) = f_*(I)$. Since $f \cdot Q_X(I) = Q_Y \cdot f(I)$ we get a morphism of functors

$$\theta : \mathbf{R}f_* \cdot \mathbf{R}Q_X \rightarrow \mathbf{R}Q_Y \cdot \mathbf{R}f_*.$$

We claim that θ is an isomorphism, i.e. $Q_Y \cdot f_*(I) \simeq \mathbf{R}Q_Y \cdot f_*(I)$. We will use a lemma.

Lemma 15.6. *The functors $\mathbf{R}f_* : D_{\text{qc}}(\text{Mod}_X) \rightarrow D_{\text{qc}}(\text{Mod}_Y)$, $\mathbf{R}f_* : D(X) \rightarrow D(Y)$, and $\mathbf{R}Q$ are way-out in both directions ([Ha]).*

Proof. Obviously all three functors are way-out left. The functor $\mathbf{R}f_* : D_{\text{qc}}(\text{Mod}_X) \rightarrow D_{\text{qc}}(\text{Mod}_Y)$ is way-out right by [Li] (see also [BoVdB], Thm.3.3.3). For the functor $\mathbf{R}Q$ see for example the proof of Proposition 1.3 in [AlJeLi].

Let us prove that the functor $\mathbf{R}f_* : D(X) \rightarrow D(Y)$ is way out right. We may assume that Y is affine and hence $f_*(-) = \Gamma(X, -)$.

Choose a finite affine open covering $\mathcal{U} = \{U_i\}_{i=1}^n$ of X . For $F \in C(X)$ denote by

$$C_{\mathcal{U}}(F) := 0 \rightarrow \oplus_{|I|=1} F_I \rightarrow \oplus_{|I|=2} F_I \rightarrow \dots$$

the corresponding (finite) Čech resolution F by alternating cochains. Here $I \subset \{1, \dots, n\}$, $i : \cap_{i \in I} U_i \rightarrow X$ and $F_I = i_* i^* F \in C(X)$. The complex F is quasi-isomorphic to $C_{\mathcal{U}}(F)$. Notice that each complex F_I is acyclic for $\Gamma(X, -)$, i.e. $\mathbf{R}\Gamma(X, F_I) = \Gamma(X, F_I)$. This shows that if F is in $D^{\leq 0}(X)$, then $\mathbf{R}f_* F \in D^{\leq n-1}(Y)$. \square

Using the lemma it suffices to prove that $\theta(M)$ is an isomorphism for a single quasi-coherent sheaf M on X ([Ha], Ch.1, Prop.7.1, (iii)). In other words we may assume that I is an (bounded below) injective resolution in Mod_X of $\phi(M)$ for $M \in Q\text{coh}X$. Then $Q_X(I)$ is an injective resolution of M in $Q\text{coh}X$. So $Q_Y \cdot f_*(I) = f_* \cdot Q_X(I)$ computes the derived direct image of M in the category of quasi-coherent sheaves. On the other hand $f_*(I)$ computes the derived direct image of $\phi(M)$. Since $f_*(I) \in D_{\text{qc}}(\text{Mod}_Y)$ it is quasi-isomorphic to $\mathbf{R}Q_Y \cdot f_*(I)$. So the needed assertion becomes $\mathbf{R}f_*(M) \simeq \mathbf{R}f_* \cdot \phi(M)$. This is proved for example in [ThTr], Appendix B, B.10. \square

Corollary 15.7. *Let $I \in C(X)$ be h-injective and $F \in C(Y)$ be h-flat. Then*

$$\text{Hom}(F, f_*(I)) = \text{Hom}_{D(X)}(F, f_*(I)).$$

Proof. An analogous statement for the category $D(\text{Mod}_X)$ is proved in [Sp].

We may assume that $I = Q_X(J)$ for an h-injective $J \in D(\text{Mod}_X)$. Then

$$\text{Hom}(F, f_* \cdot Q_X(J)) = \text{Hom}(F, Q_Y \cdot f_*(J)) = \text{Hom}(\phi(F), f_*(J)).$$

Since $\phi(F)$ is h-flat (Lemma 15.2) by [Sp] we have

$$\text{Hom}(\phi(F), f_*(J)) = \text{Hom}_{D(\text{Mod}_Y)}(\phi(F), f_*(J)),$$

and by adjunction $\text{Hom}_{D(\text{Mod}_Y)}(\phi(F), f_*(J)) = \text{Hom}_{D(X)}(F, \mathbf{R}Q_Y \cdot f_*(J))$. But in the proof of Proposition 15.5 we established a quasi-isomorphism $\mathbf{R}Q_Y \cdot f_*(J) \simeq Q_Y \cdot f_*(J)$. This proves the lemma. \square

Corollary 15.8. *The functors $\mathbf{L}f^* : D(Y) \rightarrow D(X)$ and $\mathbf{R}f_* : D(X) \rightarrow D(Y)$ are adjoint.*

Proof. It follows immediately from Corollary 15.7. \square

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